High-Performance Computing for Flows in Porous Media

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Funded by DFG within SPP 1648 (SPPEXA)

KAUST Workshop on Scalable Hierarchical Algorithms
9–11 May 2016

Challenges in Modern Computer Architectures

- Power wall
  - Power consumption is **the** limiting factor for exascale!
  - Tianhe 2: 33,8 PF (Linpack), 17.8 MW $\Rightarrow$ 500 MW
  - Clock rate stagnates but Moore's law is still valid
- Memory wall
  - Bandwidth insufficient to sustain peak performance
  - (Still) significant latency for main memory access
  - Memory access is energetically costly
  - Memory is hierarchical: register, cache, local, remote, ...
- ILP wall
  - Automatic extraction of instruction level parallelism stagnates
  - Revival of vectorization in form of SIMD instructions
- Other
  - Heterogeneous node architectures: CPU + coprocessors
  - Reduced reliability due to immense number of components?

EXA-SCALE $\Rightarrow 10^6$ **processors** $\times\ 10^3$ **threads** $\times\ 10^9$ **FLOPS**

What Does This Mean for PDE Numerics?

- Exploit three levels of parallelism:
  MPI + shared memory multithreading + SIMD instructions
- FLOPS are "for free", memory access is expensive
- Matrix-based methods
  - **Robust preconditioners** available, e.g. **AMG**
  - Poor floating point performance, bound by memory bandwidth
- Matrix-free methods
  - Explicit time-stepping or certain solvers, e.g. CG, GMG
  - Low-order FE schemes:
    - Exploit problem structure: linear transformation, regular grids, constant coefficients, "stencil codes"
    - Vectorization over several elements
  - **High-order (DG) FE schemes**: naive complexity $O(p^{\alpha d} h^{-d})$
    - Complexity reduction through **tensor-product approach**
    - Medium high order: Vectorization within one element
    - Very high order: Several threads to one element
- Other: mixed precission, increased asynchronicity

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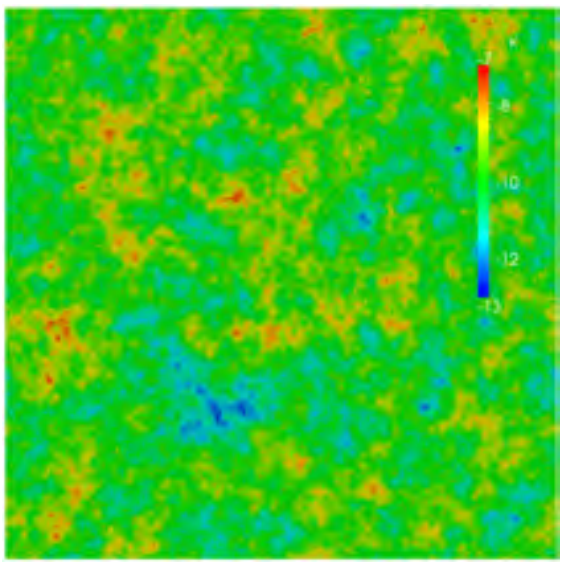
Single Phase Flow and Transport

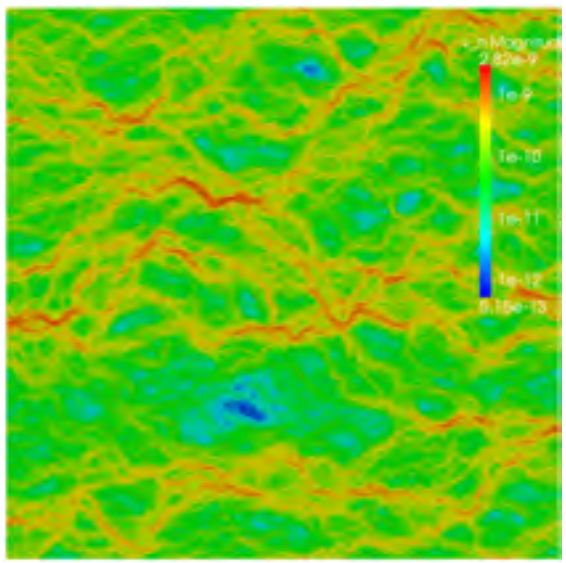
Parabolic/hyperbolic problem coupled to elliptic problem:

$$\nabla \cdot v = f, \qquad v = -\frac{K}{\mu}(\nabla p - \rho g),$$
$$\partial_t(\Phi c) + \nabla \cdot (cv - D(v)\nabla c) = q.$$

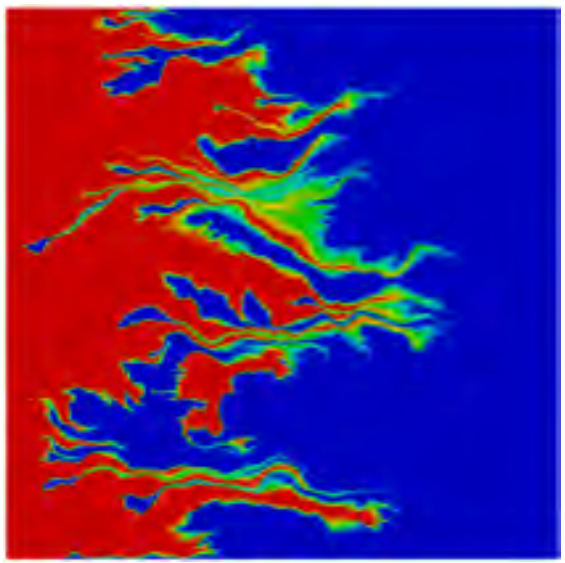
Tensor K, D, heterogeneity, convection-dominated
effective macro-scale descriptions, uncertainty of parameters



Permeability K



Velocity magnitude $\|u\|$



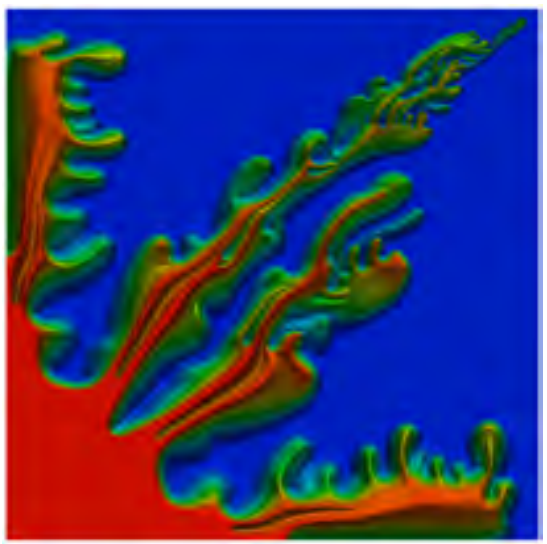
Concentration c

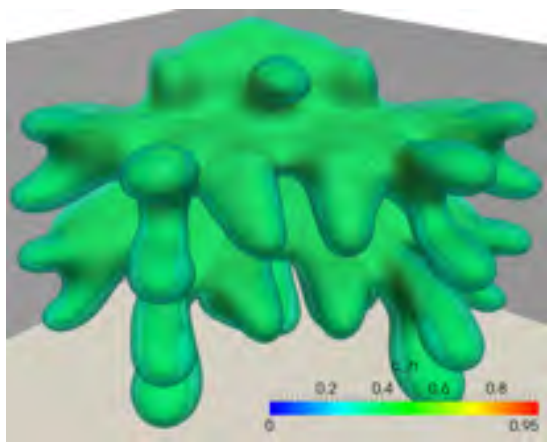
Application: Unstable Flows in Porous Media

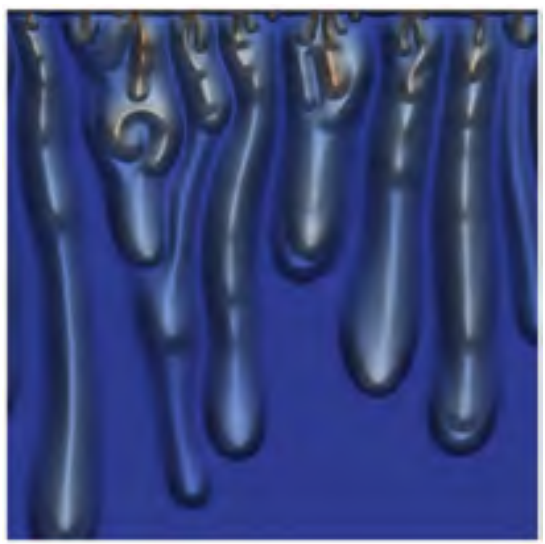
$$\nabla \cdot v = 0, \qquad v = -\frac{K}{\mu(c)}(\nabla p - \rho(c)g)$$

$$\partial_t(\Phi c) + \nabla \cdot (cv - D(v)\nabla c) = 0$$

Density driven flow, miscible discplacement
Can exploit high resolution schemes







Incompressible Immiscible Two-phase Flow

Total fluid conservation:

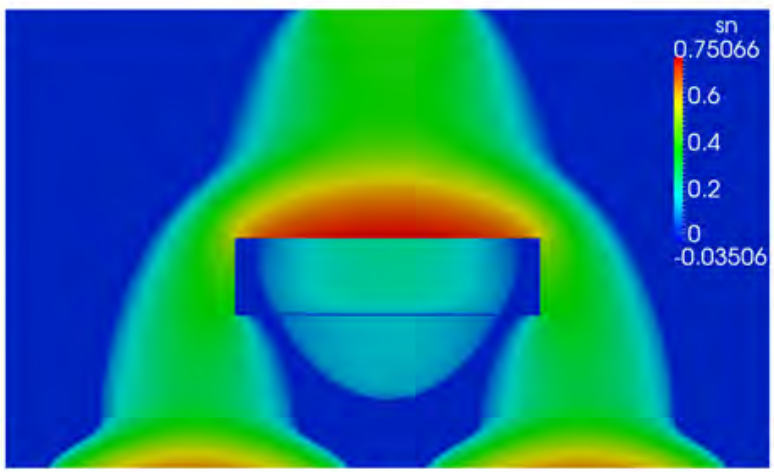
$$\nabla \cdot (v_a - \lambda_n K \nabla \phi_c) = q, \qquad v_a = -\lambda_t K \nabla \phi_w.$$

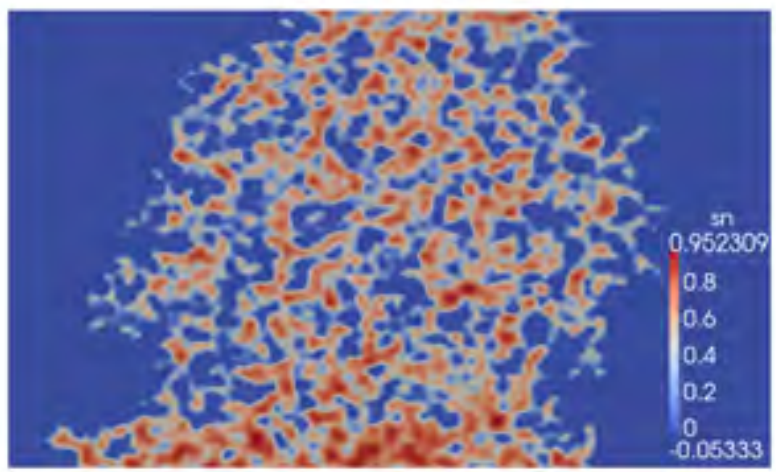
Conservation of non-wetting phase:

$$\Phi \partial_t (1 - \psi(\phi_c)) + \nabla \cdot (f_n(1 - \psi(\phi_c)) v_a - \lambda_n (1 - \psi(\phi_c)) K \nabla \phi_c) = q_n.$$

Primary variables: $\phi_w = p_w - \rho_w g d$, $\phi_c = p_c - (\rho_n - \rho_w) g d$

$\psi(\phi_c)$, $\lambda_n(\phi_c)$, $\lambda_t(\phi_c)$, $f_n(\phi_c)$ are nonlinear functions





Discontinuous Galerkin (DG) Finite Element Methods

- Advantages
  - Potentially high order convergence
  - High-order methods beneficial even with low regularity
  - Element-wise conservative
  - Full permeability tensors, discontinuous coefficients
  - Handles elliptic, parabolic and hyperbolic equations equally well
  - Unstructured grids, nonconforming refinement
  - Contiguous memory access pattern
  - Block matrix structure: less indirect access, BLAS level 3

- Disadvantages
  - No monotonicity (for 2nd order problems)
  - Free parameter to tune
  - Higher number of DOFs compared to standard FE or FV
  - Linear systems considered to be more difficult to solve

DG for Groundwater Flow Equation

SIPG + weighted averages (*Di Pietro, Ern, Guermond '2008*):

$$a_h(p,w;c) = \sum_{T\in\mathcal{T}_h}\int_T [\frac{K}{\mu(c)}(\nabla p - \rho(c)g)]\cdot\nabla w - \sum_{F\in\mathcal{F}_h^{iD}}\int_F \nu_F\cdot\{\frac{K}{\mu(c)}(\nabla p-\rho(c)g)\}_\omega [\![w]\!]$$

$$- \sum_{F\in\mathcal{F}_h^{iD}}\int_F \theta\nu_F\cdot\{\frac{K}{\mu(c)}\nabla w\}_\omega [\![p]\!] + \sum_{F\in\mathcal{F}_h^{iD}}\int_F \gamma_{F,f}[\![w]\!][\![p]\!]$$

$$l(w) = \sum_{T\in\mathcal{T}_h}\int_T q_f w - \sum_{F\in\mathcal{F}_h^{N}}\int_F j_f w \qquad \theta = 1 \text{ (SIPG)},\ \theta = 0 \text{ (OBB)},\ \theta = -1 \text{ (NIPG)}$$

$$\delta_{Kn}^{\pm} = \nu_F^t K^{\pm}\nu_F/\mu(c)^{\pm},\quad \omega^- = \frac{\delta_{Kn}^+}{\delta_{Kn}^- + \delta_{Kn}^+},\quad \omega^+ = \frac{\delta_{Kn}^-}{\delta_{Kn}^- + \delta_{Kn}^+}.$$

Penalty term ($\langle\cdot,\cdot\rangle$ harmonic average, p polynomial degree, d space dimension):

$$\gamma_{F,f} = \alpha\langle\delta_{Kn}^-,\delta_{Kn}^+\rangle M,\quad M = p(p+d-1)\frac{|F|}{\min(|T^-(F)|,|T^+(F)|)}$$

Contents

Multigrid Method

- Wish to solve large $Ax = b$
- Assume hierarchy of matrices

$$A_l \in \mathbb{R}^{I_l \times I_l} \qquad l = 0, \ldots, L, \; A_L = A$$

  $I_l = \{1, \ldots, n_l\}$ denotes the index set on level l
- Transfer matrices

$$R_l \in \mathbb{R}^{I_{l-1} \times I_l}, \; P_l \in \mathbb{R}^{I_l \times I_{l-1}} \qquad l = 1, \ldots, L$$

- Geometric multigrid: A_l, R_l, P_l related to hierarchy of finite element spaces

$$V_l = V(\mathcal{T}_l) = \text{span}\{\varphi_{i,l} : i \in I_l\} \qquad l = 0, \ldots, L$$

  constructed from a mesh hierarchy

$$\mathcal{T}_l \qquad l = 0, \ldots, L$$

- Algebraic multigrid: Only A and b are given

Multiplicative Multigrid V-Cycle

Require: x approximation of solution of linear system $Ax = b$
Ensure: x improved approximation

$d_L \leftarrow b - Ax$; $v_L \leftarrow 0$; ▷ Compute defect
for $l \in \{L, \ldots, 1\}$ **do**
  Smooth(v_l, d_l, ν_1); ▷ Pre-smoothing (updates v_l)
  $d \leftarrow d_l - A_l v_l$. ▷ Compute defect
  $d_{l-1} \leftarrow R_l d$; ▷ Restrict defect
  $v_{l-1} \leftarrow 0$; ▷ Prepare coarser level
end for
Solve $A_0 v_0 = d_0$; ▷ Exact coarsest level solve
for $l \in \{1, \ldots, L\}$ **do**
  $v_l \leftarrow v_l + \xi P_l v_{l-1}$; ▷ Prolongate coarse grid correction
  Smooth(v_l, d_l, ν_2); ▷ Post-smoothing (updates v_l)
end for
$x \leftarrow x + v_L$;

Geometric and Algebraic Multigrid

- **Geometric multigrid**
  - $\mathcal{T}_l, 0 \leq l \leq L$: nested sequence of meshes
  - $V_l = V(\mathcal{T}_l)$ nested sequence of finite element spaces
  - Basis representation: $\phi_{i,l-1} = \sum_{j \in I_l} (R_l)_{i,j} \phi_{j,l}$ gives R_l
  - Then $A_{l-1} = R_l A_l P_l$ with $P_l = R_l^T$ (Galerkin product)
- **Ruge-Stüben algebraic multigrid**
  - Partition index set $I_l = F_l \cup C_l$ and set $I_{l-1} = C_l$
  - Determine entries of prolongation matrix $P_l \in \mathbb{R}^{I_l \times C_l}$
  - Set $A_{l-1} = R_l A_l P_l$ with $R_l = P_l^T$
  - Selection of C_l and interpolation factors depend on A_l
- **Agglomeration algebraic multigrid**
  - Partition index set $I_l = \bigcup_{i=1}^{n_{l-1}} C_{i,l}$ (clustering)
  - Partitioning depends on A_l
  - Set $(R_l)_{i,j} = 1$ if $j \in C_{i,l}$ and zero else ("piecewise constants")
  - Set $A_{l-1} = R_l A_l P_l$ with $P_l = R_l^T$
  - Important: scale coarse grid correction by factor ξ ($\xi \approx 2$ for regular 2^d coarsening of diffusion problem)

Agglomeration AMG: Details I

- References
  - Developed independently in the mid 90s: *Vanek et al (1994)*, *Braess (1995)*, *Raw et al (1996)*
  - Theory and practice by Notay, e.g. *Notay, Napov, JCP, 2015*
  - Combinatorial algebraic multigrid: I. Koutis, G.L. Miller, and R. Peng. arXiv:1102.4842v4
- Our implementation follows *Raw et al (1996)*
  - Symmetric strength of connection criterion (for M-matrices): Edge $(i,j) \in I_l \times I_l$ is called strong iff

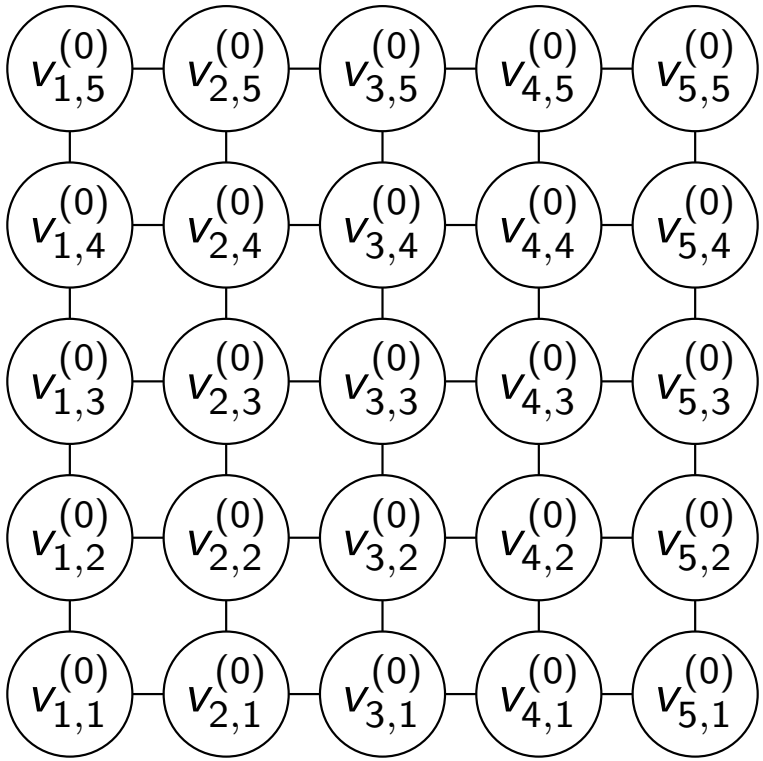
$$\frac{a_{ij}\,a_{ji}}{a_{ii}\,a_{jj}} \geq \delta \min(\eta_i, \eta_j)$$

    with $\eta_i = \max_j \frac{a_{ij}\,a_{ji}}{a_{ii}\,a_{jj}}$ and threshold $0 < \delta < 1$.
  - Greedy region growing algorithm builds clusters of arbitrary size (aggressive coarsening)
  - Fill-in minimization: roundness, merge small clusters
  - Non-M-matrices: ignore positive off-diagonals
  - Systems of PDEs: norm of blocks, guiding component

Agglomeration AMG: Details II

- Parallelization
  - Row-wise partitioning of the matrix with 1-overlap
  - Build clusters in parallel and exchange overlap
  - Build coarse system in parallel w/o communication
  - $P \gtrapprox 5000$ requires successive coarsening
- Advantages
  - Quite robust for elliptic problems with heterogeneous coefficients
  - Sparsity pattern is very well preserved on coarser levels
  - Conjecture: AAMG V-Cycle preconditioner yields $O(L)$ condition number
- Disadvantages
  - Needs to be used as preconditioner
  - Convergence for Poisson equation is generally not optimal (is cured by smoothed aggregation)

Sequential Agglomeration



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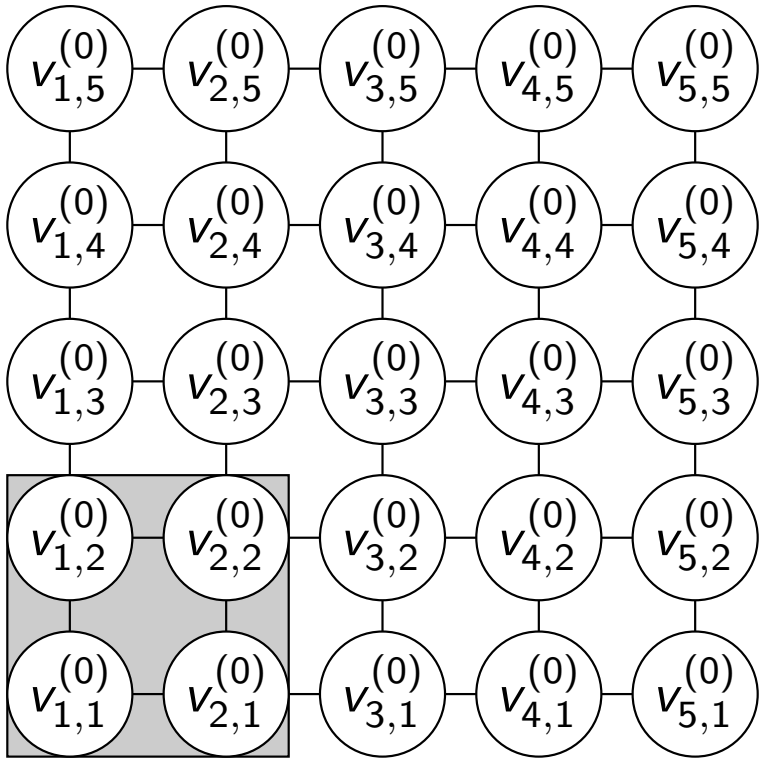

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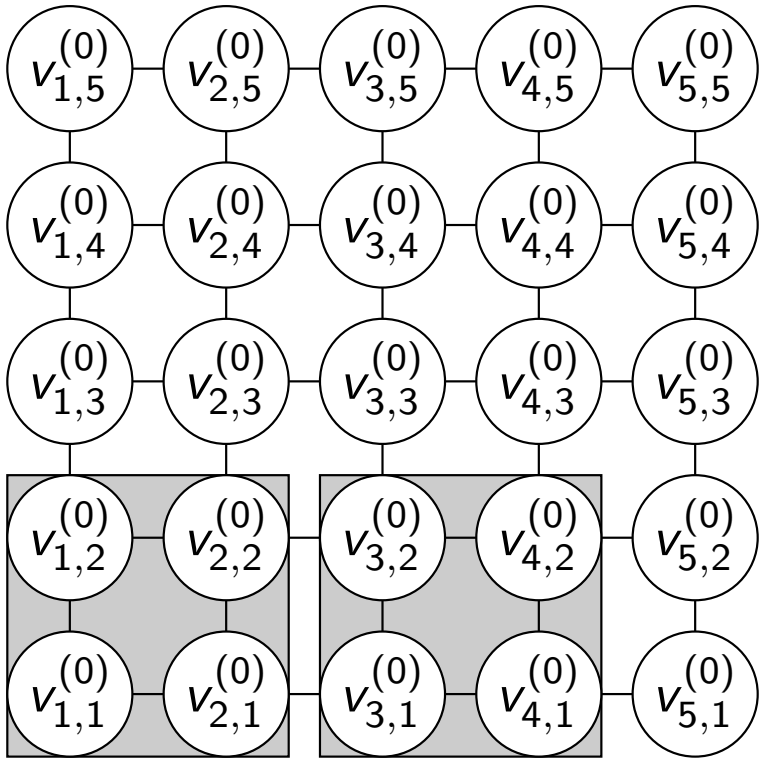

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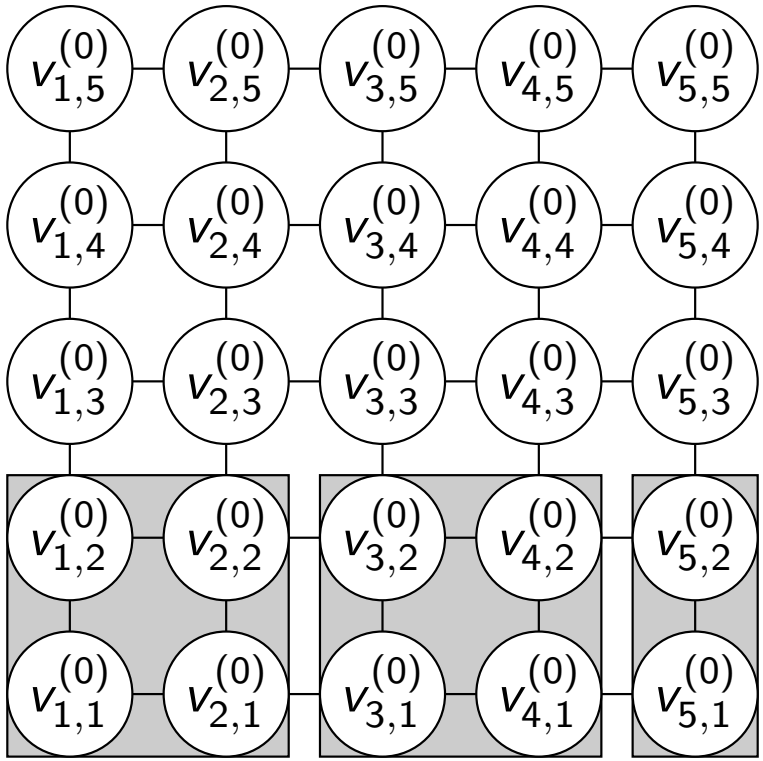

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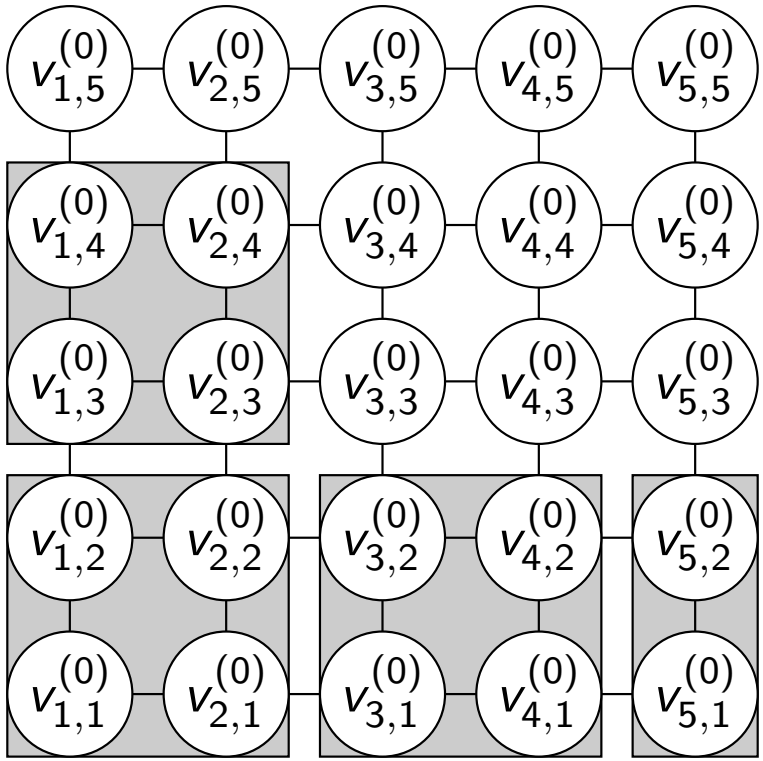

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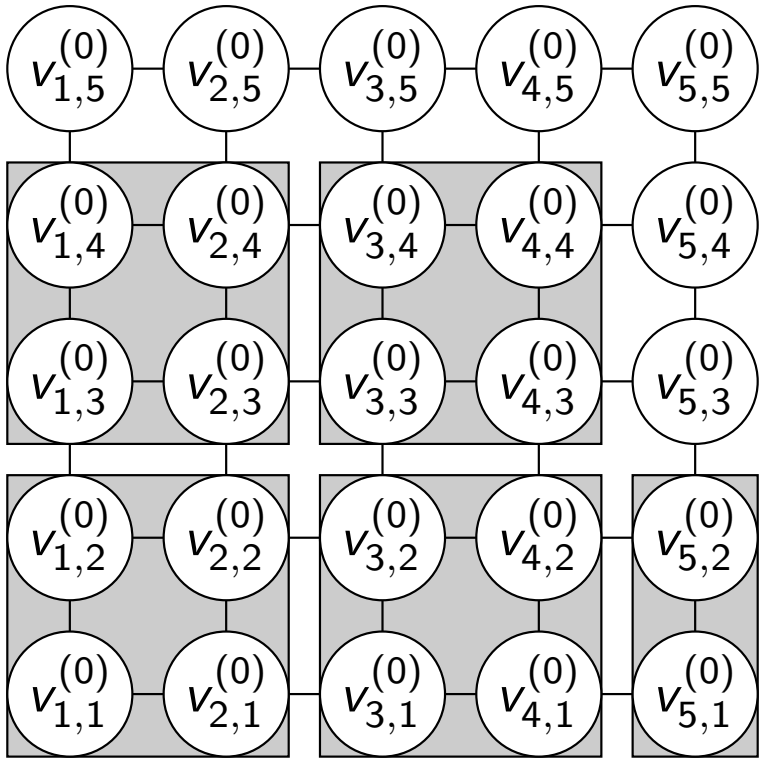

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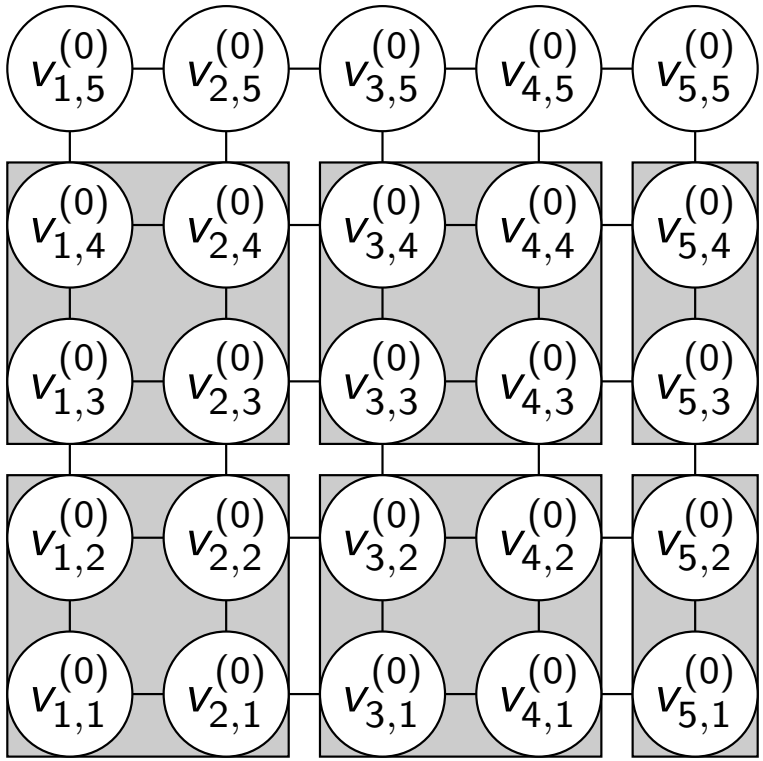

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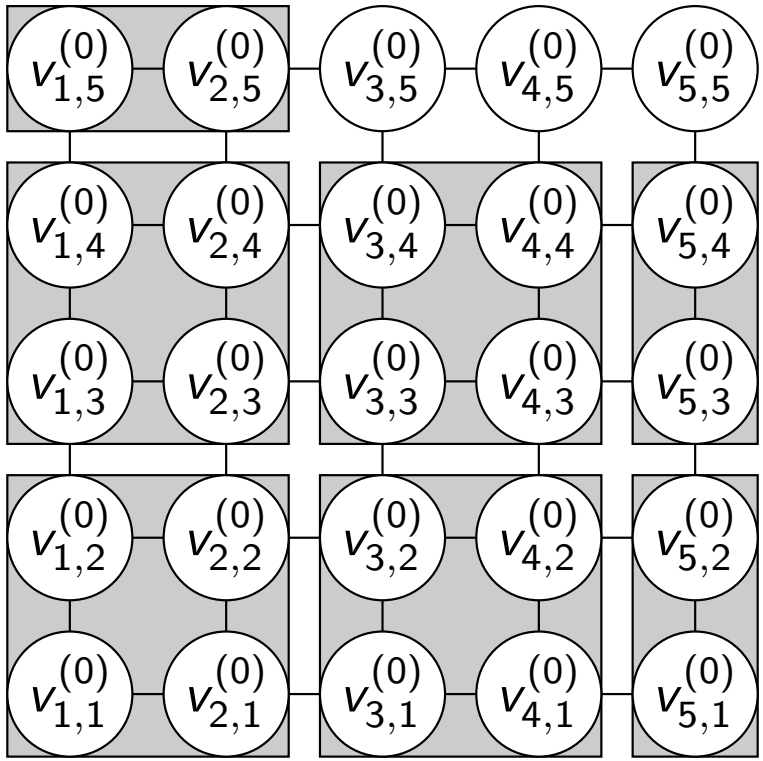

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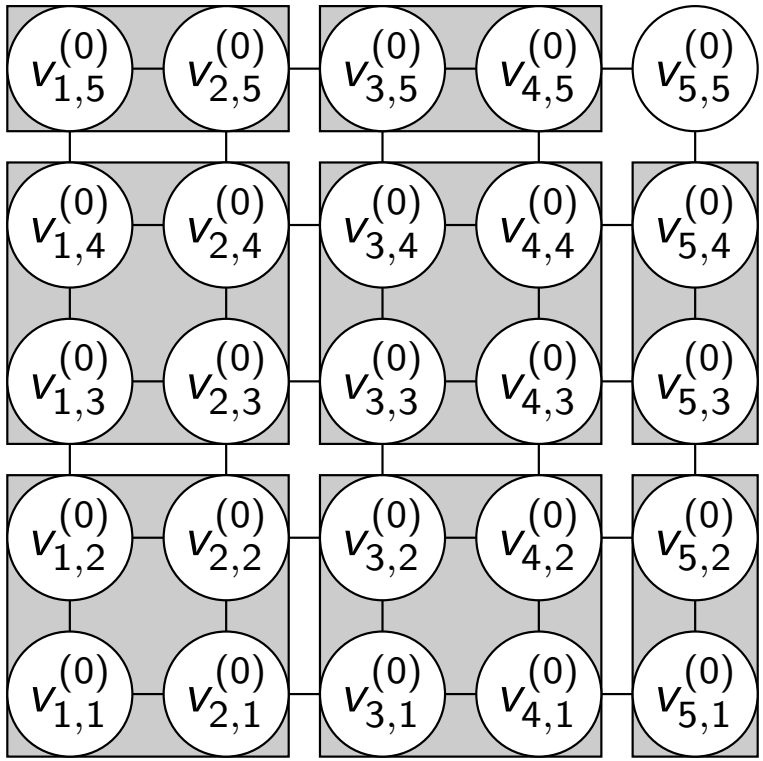

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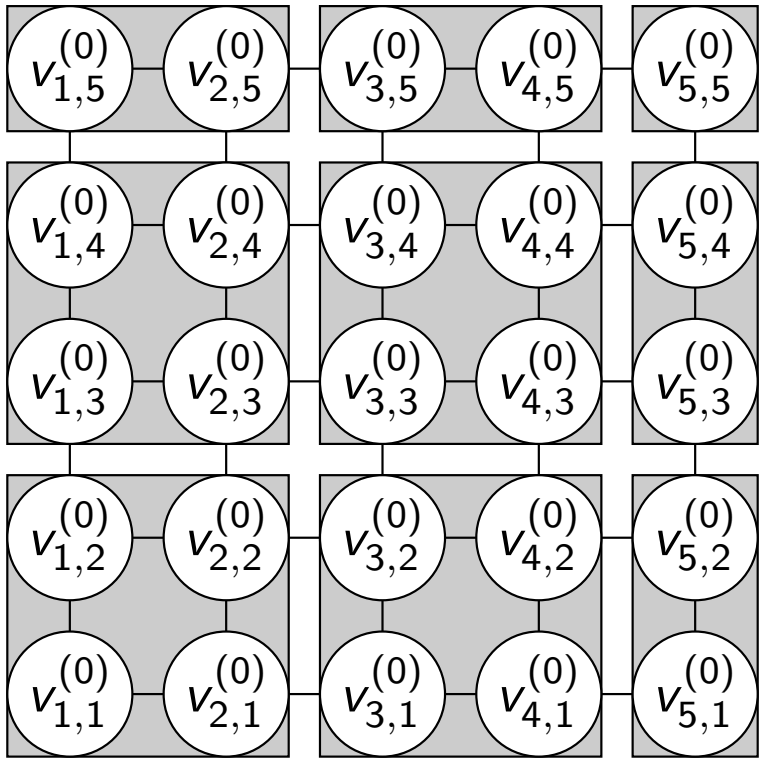

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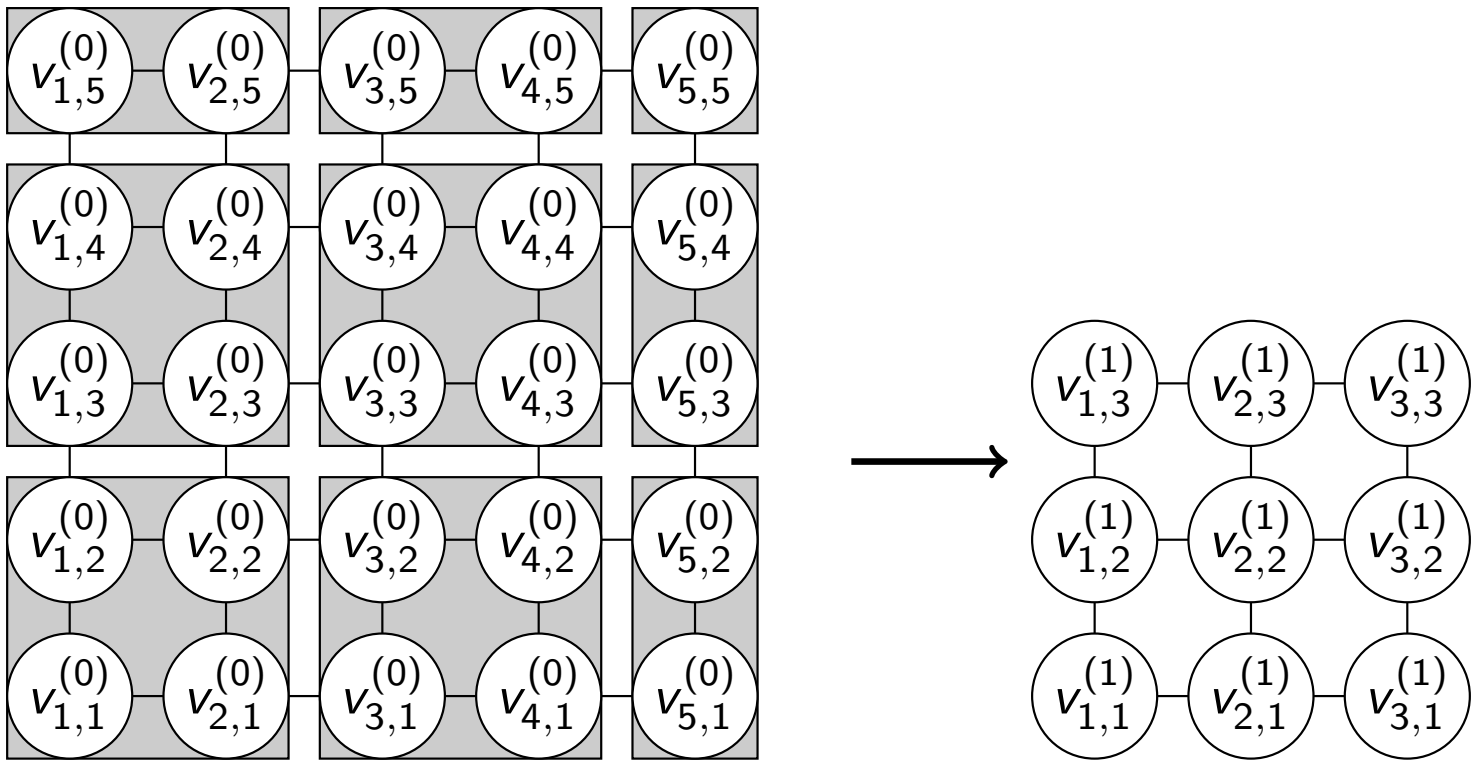
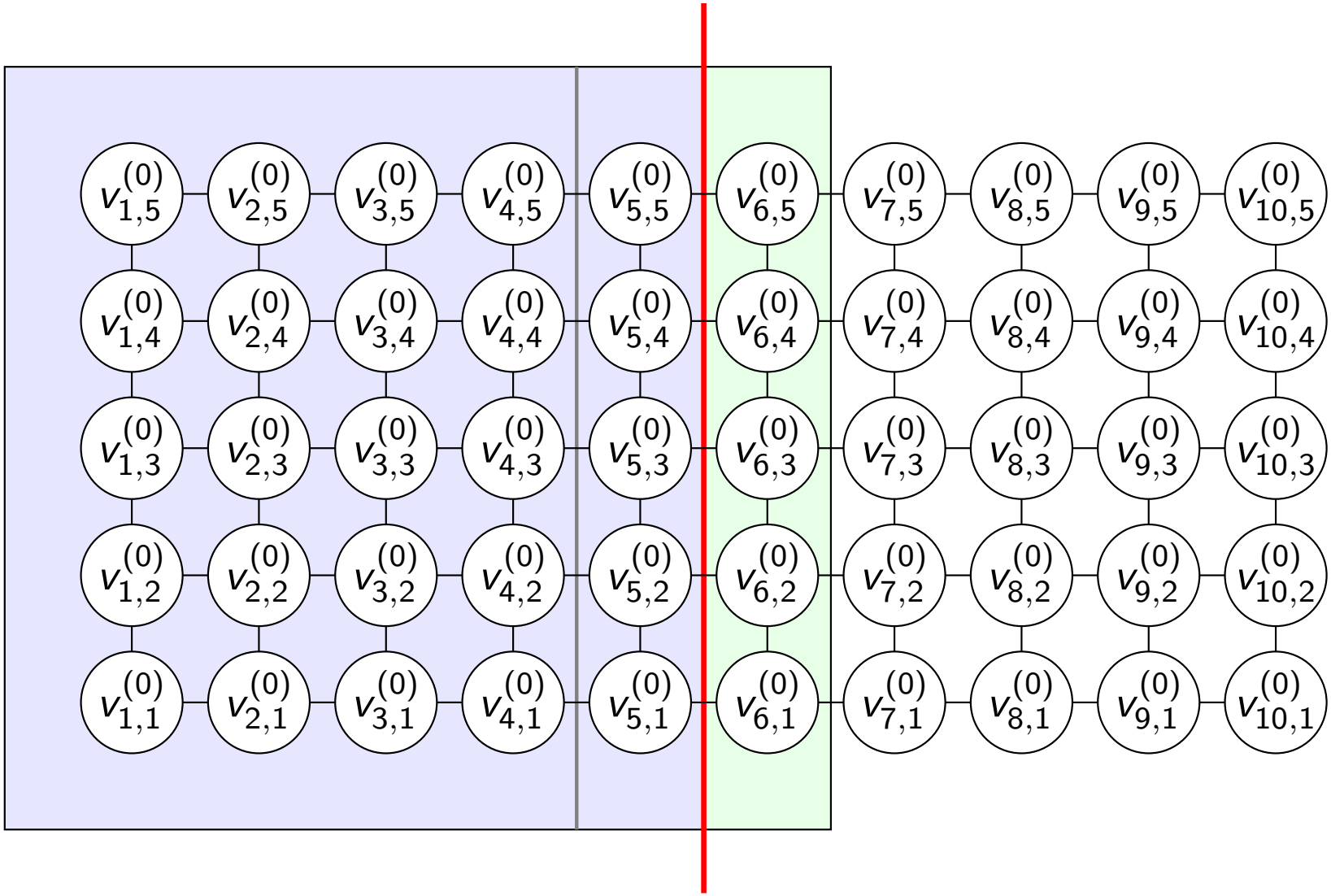


Parallel Agglomeration



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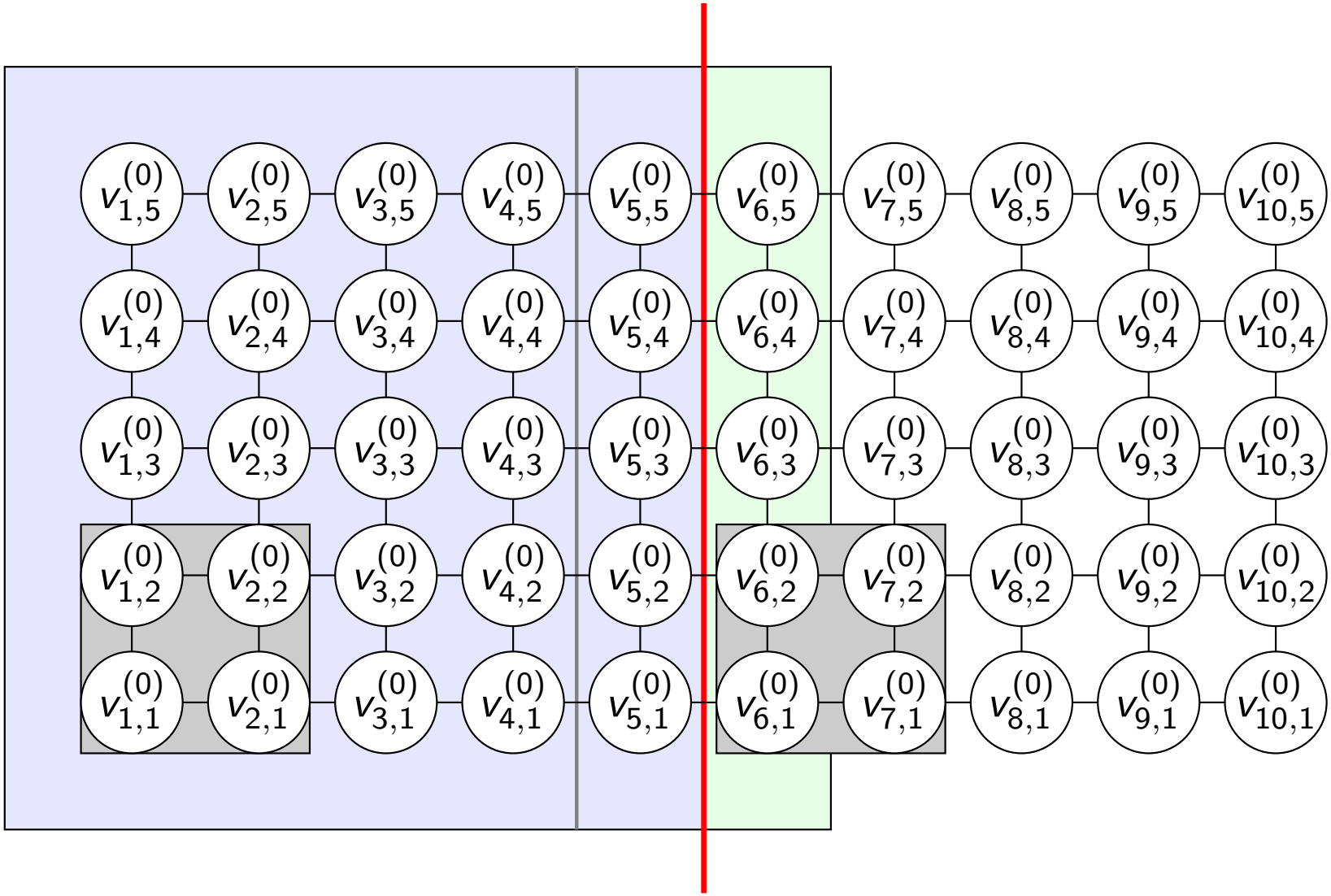

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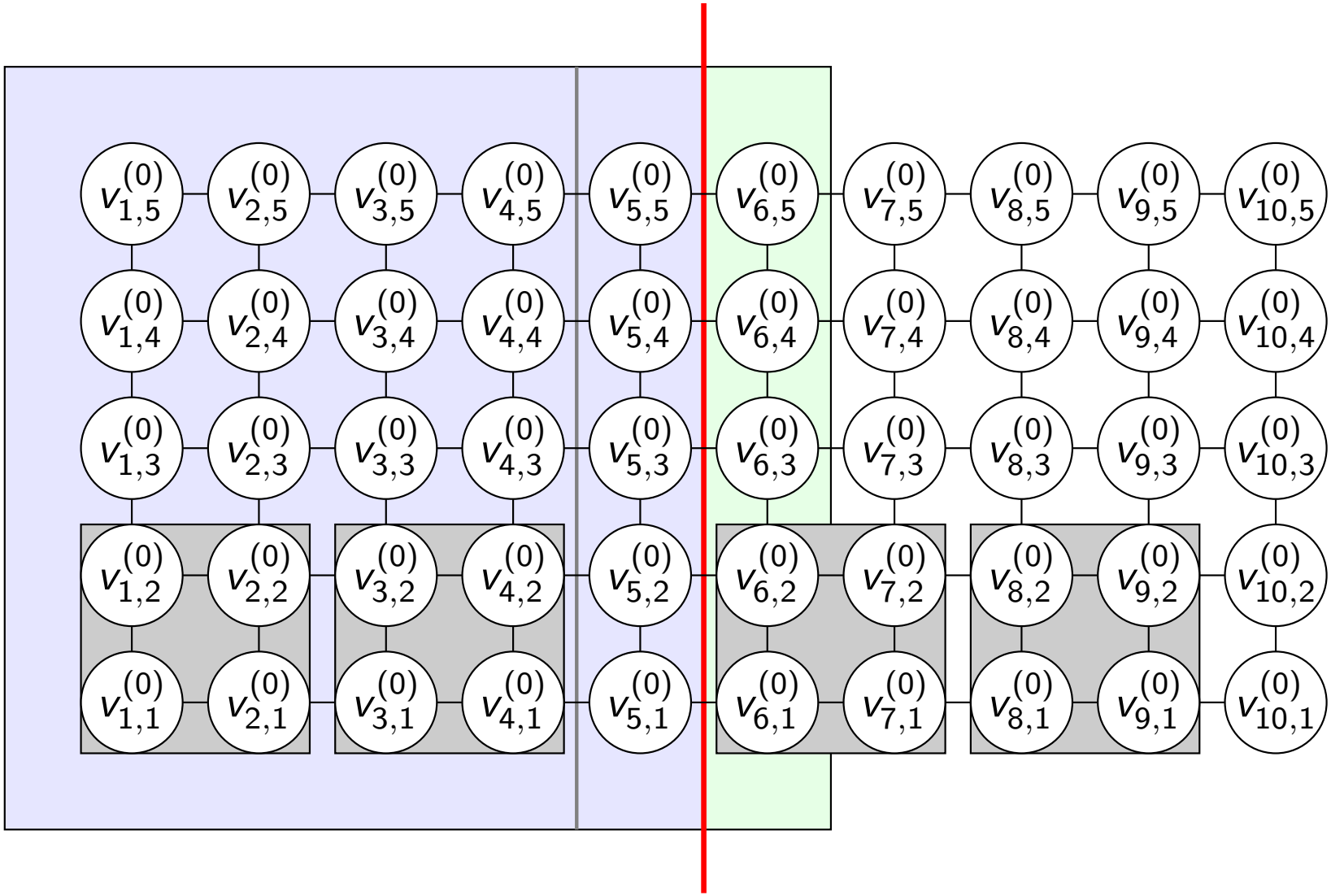

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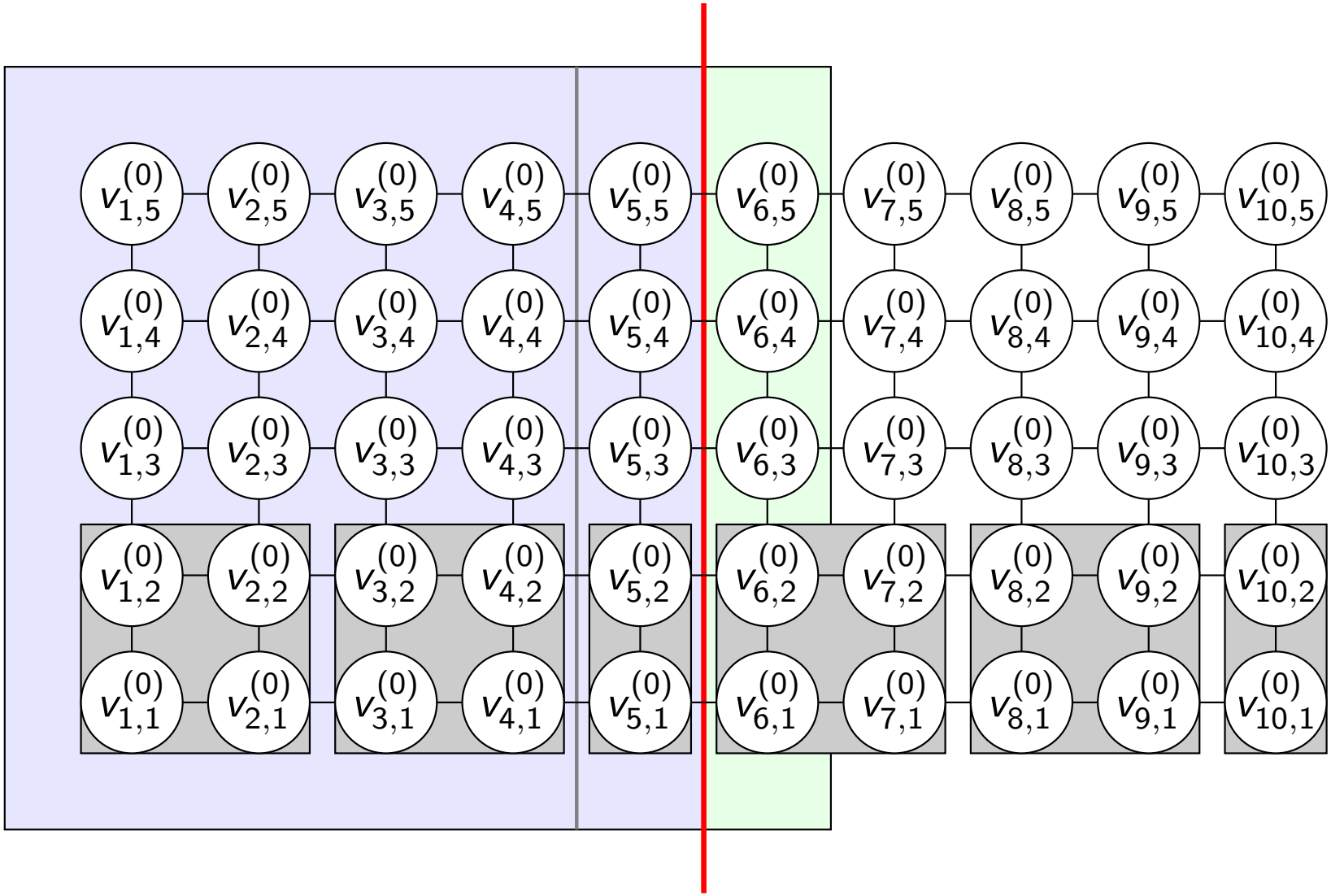

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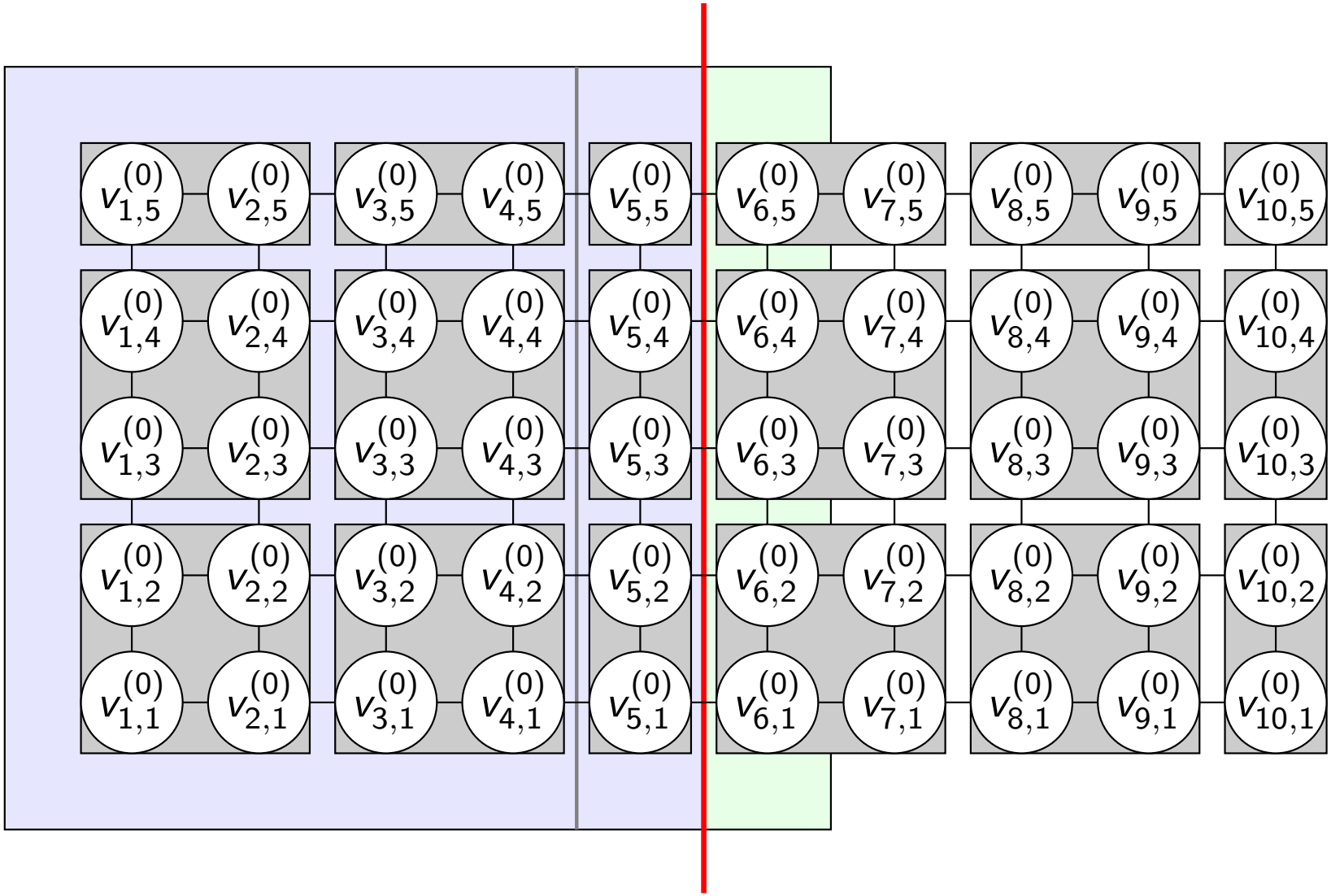

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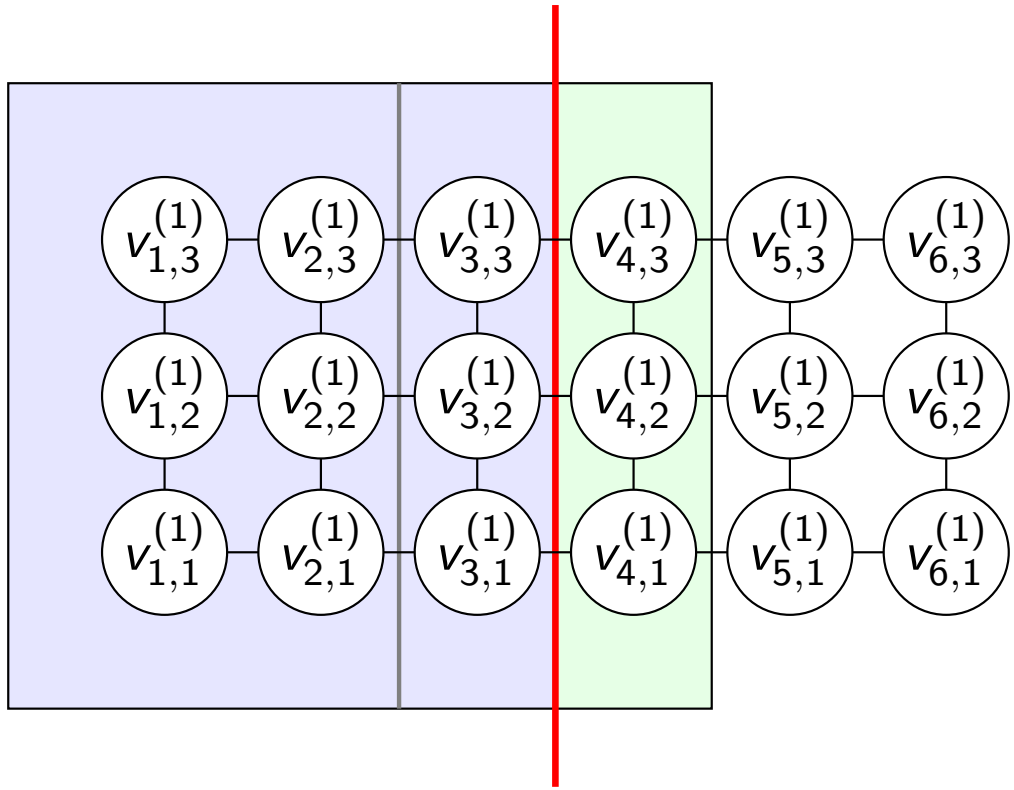


AMG Weak Scaling Results

- AAMG and results presented is work of **Markus Blatt**
- BlueGene/P at Jülich Supercomputing Center
- $P \cdot 80^3$ degrees of freedom (5120^3 finest mesh), CCFV
- Poisson problem, 10^{-8} reduction
- AMG used as preconditioner in BiCGStab (2 V-Cycles!)
- **Scaling starts at 1**

procs	1/h	lev.	TB	TS	NIT	TIT	TT
1	80	5	19.86	31.91	8	3.989	51.77
8	160	6	27.7	46.4	10	4.64	74.2
64	320	7	74.1	49.3	10	4.93	123
512	640	8	76.91	60.2	12	5.017	137.1
4096	1280	10	81.31	64.45	13	4.958	145.8
32768	2560	11	92.75	65.55	13	5.042	158.3
262144	5120	12	188.5	67.66	13	5.205	256.2

Multiplicative AMG-DG Preconditioner

Require: x_F approximation of solution of DG system $A_F x_F = b_F$

Ensure: x_F improved approximation

$d_F \leftarrow b_F - A_F x_F$; $v_F \leftarrow 0$; ▷ Compute defect

Smooth(v_F, d_F, ν_1); ▷ Pre-smoothing on DG system

$d_C \leftarrow R_F(d_F - A_F v_F)$; ▷ Restrict defect to subspace

$v_C \leftarrow 0$; ▷ Prepare coarse level

AMG(v_C, d_C); ▷ Apply AMG V-Cycle

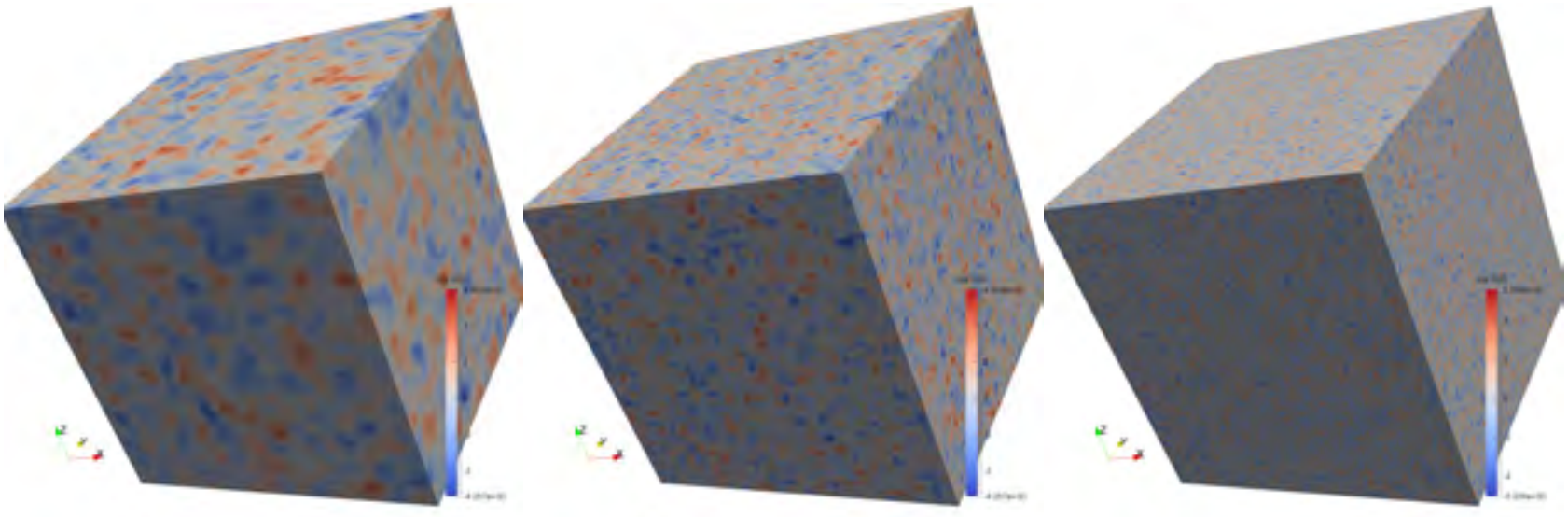
$v_F \leftarrow v_F + R_F^T v_C$ ▷ Add coarse grid correction

Smooth(v_F, d_F, ν_2); ▷ Post-smoothing on DG system

$x_F \leftarrow x_F + v_F$; ▷ Provide result

- R_F obtained from embedding $V_C \subset V_F = V_{DG}$
- V_C may be Q_1 conforming or P_0
- Method is not purely algebraic
- Smoothing step takes majority of computing time
- Smoothing step may be implemented matrix-free
- See *B., Blatt, Scheichl, NLAA, 2012*

AMG Weak Scaling Results: Heterogeneous Problems



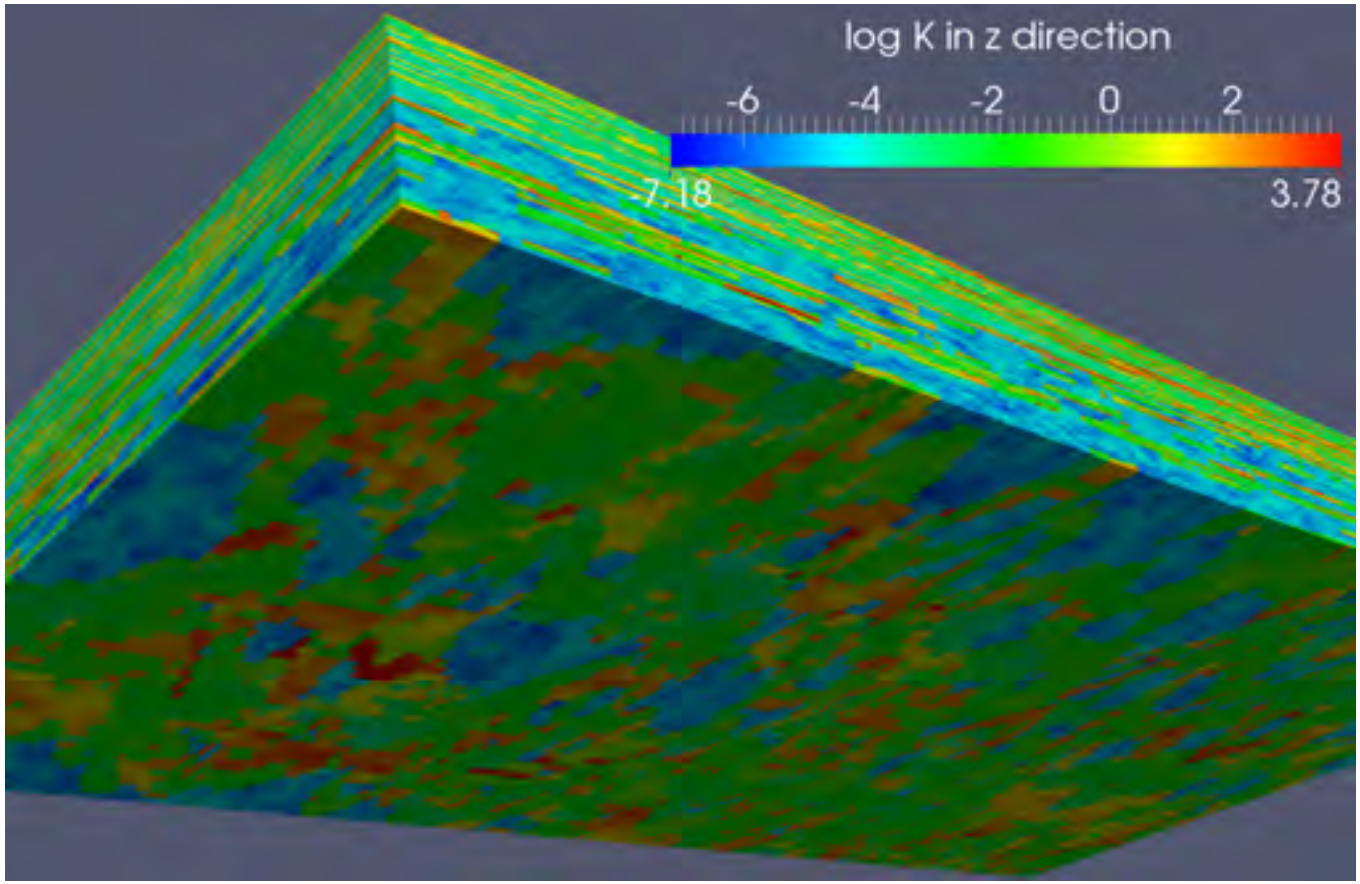
- Gaussian random field with Gaussian covariance function
- Picture: correlation length $2h$, variance 5
- 10 orders of magnitude variation
- Problem gets more difficult with refinement

AMG Weak Scaling Results: Hybrid DG-AMG

Problem	P	TIT	NIT	TIT	NIT
		Q1		DG1/Q1	
CL=$20h$	1	0.24	10	3.90	9
VAR=1	8	0.32	14	4.95	12
	64	0.35	16	5.19	15
	512	0.37	17	5.24	18
	4096	0.61	19	5.88	18
		CCFV		DG1/P0	
CL=$2h$	1	0.08	18	2.41	32
VAR=5	8	0.12	25	3.11	49
	64	0.14	25	3.30	55
	512	0.15	40	3.36	74
	4096	0.75	50	3.92	94

- $60^3 = 216000$ cells per core, 10^{-8} reduction
- Xeon CPU E5-2630 v3 @ 2.40GHz "Haswell", 8 cores
- 416 nodes, 2 CPUs per node

SPE10 Test Problem



- Domain $1200 \times 2200 \times 170$ ft^3
- $60 \times 220 \times 85$ mesh, 1.1 Mio cells
- Anisotropic mesh, anisotropic diagonal permeability tensor
- 11 orders of magnitude variation

SPE10 Results

- $-\nabla \cdot (K \nabla u) = 0$, Dirichlet on sides, Neumann top/bottom
- 10^{-6} reduction from zero initial value

Strong Scaling

		# cores			
Scheme	What	1	4	16	80
CCFV	NIT	23	24	27	27
	TIT	0.27891	0.08955	0.02714	0.00907
	Overall Speedup		3.0	8.8	26.1
DG1/P0	NIT	35	38	37	42
	TIT	10.058	3.026	1.001	0.289
	Overall Speedup		3.1	9.5	29.0

Iteration numbers for h refinement, $P = 80$

# cells	$1.1 \cdot 10^6$	$8.8 \cdot 10^6$	$70.4 \cdot 10^6$
CCFV	27	33	39
DG1/P0	42	54	66

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Exploiting Tensor Product Structure

Tensor product basis: $\phi_i(x) = \theta_{i_d}(x_d) \dots \theta_{i_1}(x_1)$

Tensor product quadrature: $\Xi_j = (\xi_{j_1}, \dots, \xi_{j_d})^T$

where $i = (i_1, \dots, i_d) \in I^d$ and $j = (j_1, \dots, j_d) \in J^d$

Evaluate finite element function on ref elem at quadrature points:

$$
\begin{aligned}
u_h(\Xi_i) &= \sum_{j \in J^d} c_j \phi_j(\Xi_i) && (i \in I^d) \\
&= \sum_{j_d \in J} \dots \sum_{j_1 \in J} \theta_{j_d}(\xi_{i_d}) \dots \theta_{j_1}(\xi_{i_1}) \; c_{j_1,\dots,j_d} && \text{(tensor product struct.)} \\
&= \sum_{j_d \in J} \dots \sum_{j_1 \in J} A^{(d)}_{i_d,j_d} \dots A^{(1)}_{i_1,j_1} \; c_{j_1,\dots,j_d} && (A^{(k)} \in \mathbb{R}^{m \times n})
\end{aligned}
$$

Evaluation at all quadrature points is matrix-vector product:

$$y = Ax = (A^{(d)} \otimes \dots \otimes A^{(1)})\, x, \qquad A_{(i_1,\dots,i_d),(j_1,\dots,j_d)} = \Pi_{k=1}^{d} A^{(k)}_{i_k,j_k}$$

Same structure obtained for BLF evaluation: $a(u_h, \phi_i)$, $i \in I^d$

Sum Factorization

Extract common factors:

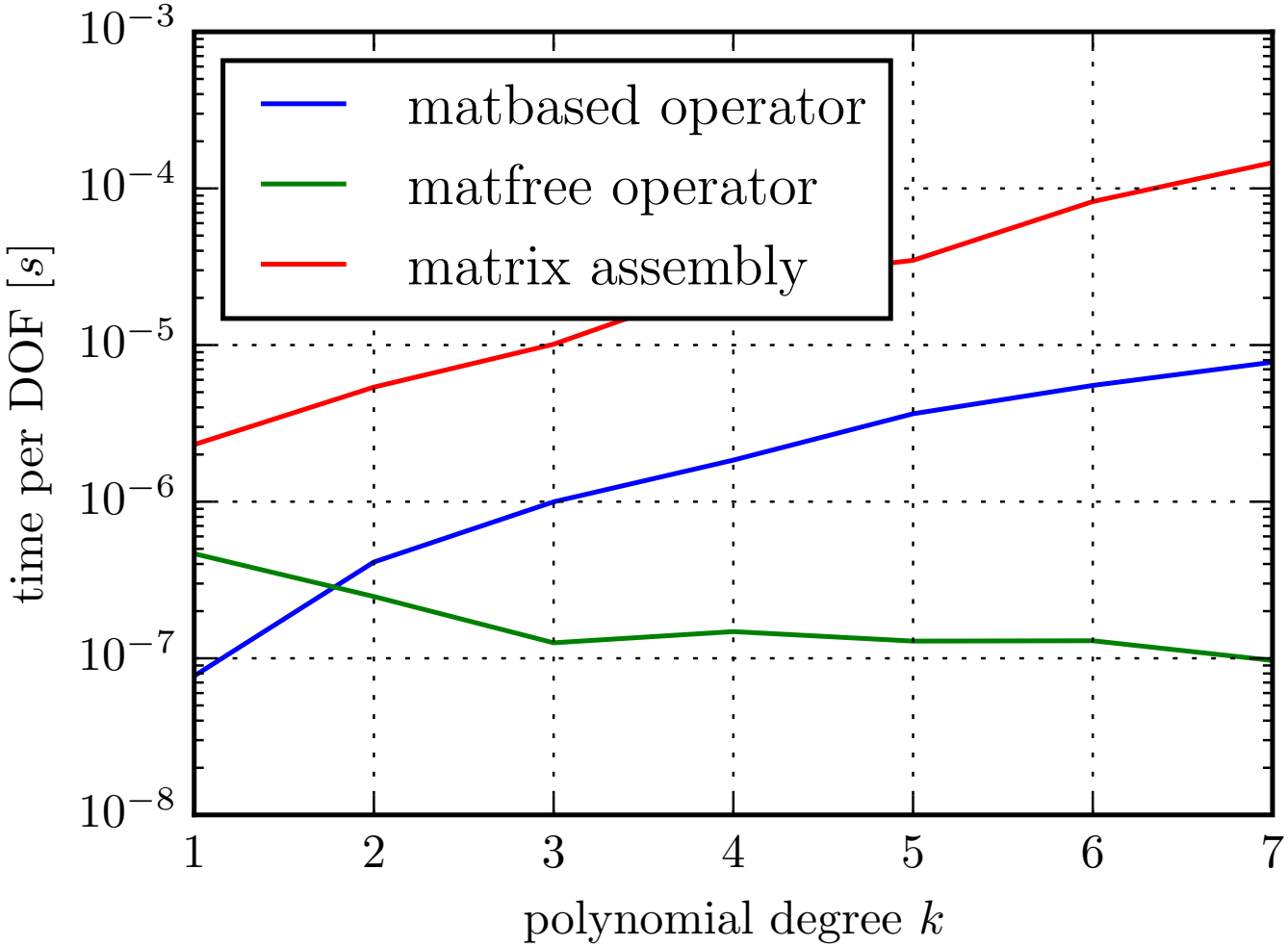
$$
\begin{aligned}
y_{i_1,\ldots,i_d} &= \sum_{j_d \in J} \cdots \sum_{j_1 \in J} A^{(d)}_{i_d,j_d} \cdots A^{(1)}_{i_1,j_1} x^{(0)}_{j_1,\ldots,j_d} \\
&= \sum_{j_d \in J} \cdots \sum_{j_2 \in J} A^{(d)}_{i_d,j_d} \cdots A^{(2)}_{i_2,j_2} \underbrace{\left(\sum_{j_1 \in J} A^{(1)}_{i_1,j_1} x^{(0)}_{j_1,\ldots,j_d} \right)}_{x^{(1)}_{i_1,j_2,\ldots,j_d}} \\
&= \sum_{j_d \in J} \cdots \sum_{j_3 \in J} A^{(d)}_{i_d,j_d} \cdots A^{(3)}_{i_3,j_3} \underbrace{\left(\sum_{j_2 \in J} A^{(2)}_{i_2,j_2} x^{(1)}_{i_1,j_2,\ldots,j_d} \right)}_{x^{(2)}_{i_2,i_1,j_3,\ldots,j_d}} \\
&= \ldots = \sum_{j_d \in J} A^{(d)}_{i_d,j_d} x^{(d-1)}_{i_{d-1},\ldots,i_1,j_d}
\end{aligned}
$$

Do this for all $(i_1, \ldots, i_d)$ simultaneously

Sum Factorization for Finite Elements

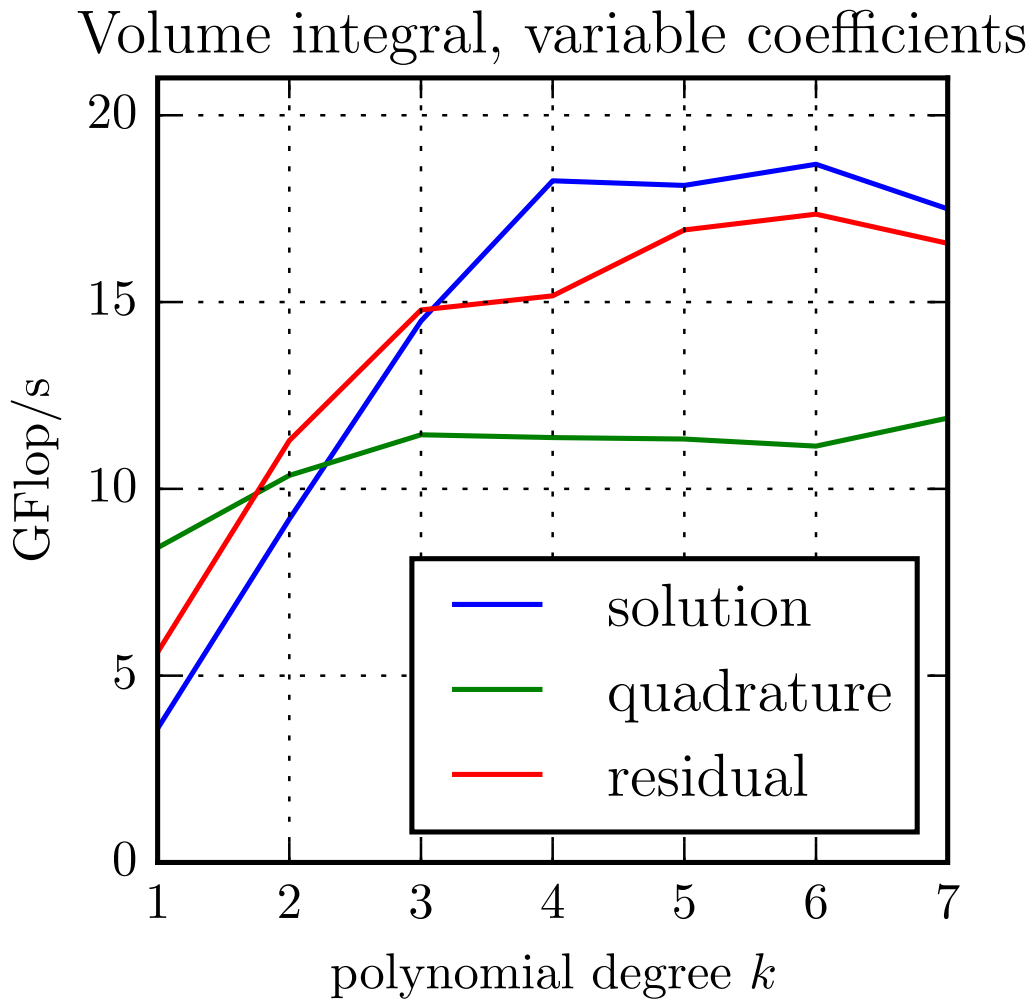
- Sum factorization algorithm by *Buis, Dyksen (1996)*:
- For $k = 1, \dots d$ do („*dim by dim approach*")
  - compute **matrix-matrix product** $X^{(k)} = A^{(k)} X^{(k-1)}$
  - "rotate" $X^{(k)}$ to make j_{k+1} the row index (transposition in 2d)
- Complexity reduction ($n = m = p + 1$):
  - $O(p^{2d}) \rightarrow O(p^{d+1})$ for operator application (mat-vec)
  - $O(p^{3d}) \rightarrow \ O(p^{2d+1})$ for Jacobian assembly
- Small matrices, e.g. $(4 \times 4) \cdot (4 \times 16)$ for $\mathbb{Q}_3$ in 3d
- FE operator evaluation, by e.g. *Melenk, Gerdes, Schwab (2001)*, *Kronbichler, Korman (2012)*:
  1. compute $\hat{u}_h$, $\nabla \hat{u}_h$ at quadrature points, $O(p^{d+1})$
  2. compute coefficients, geometry factors at quad. points, $O(p^d)$
  3. compute residuals $O(p^{d+1})$
- Applicable to nonlinear problems
- Applicable to non-affine transformations
- In DG face terms are dominating: $O(p^d)$ complexity observed

Time per DOF Single Core



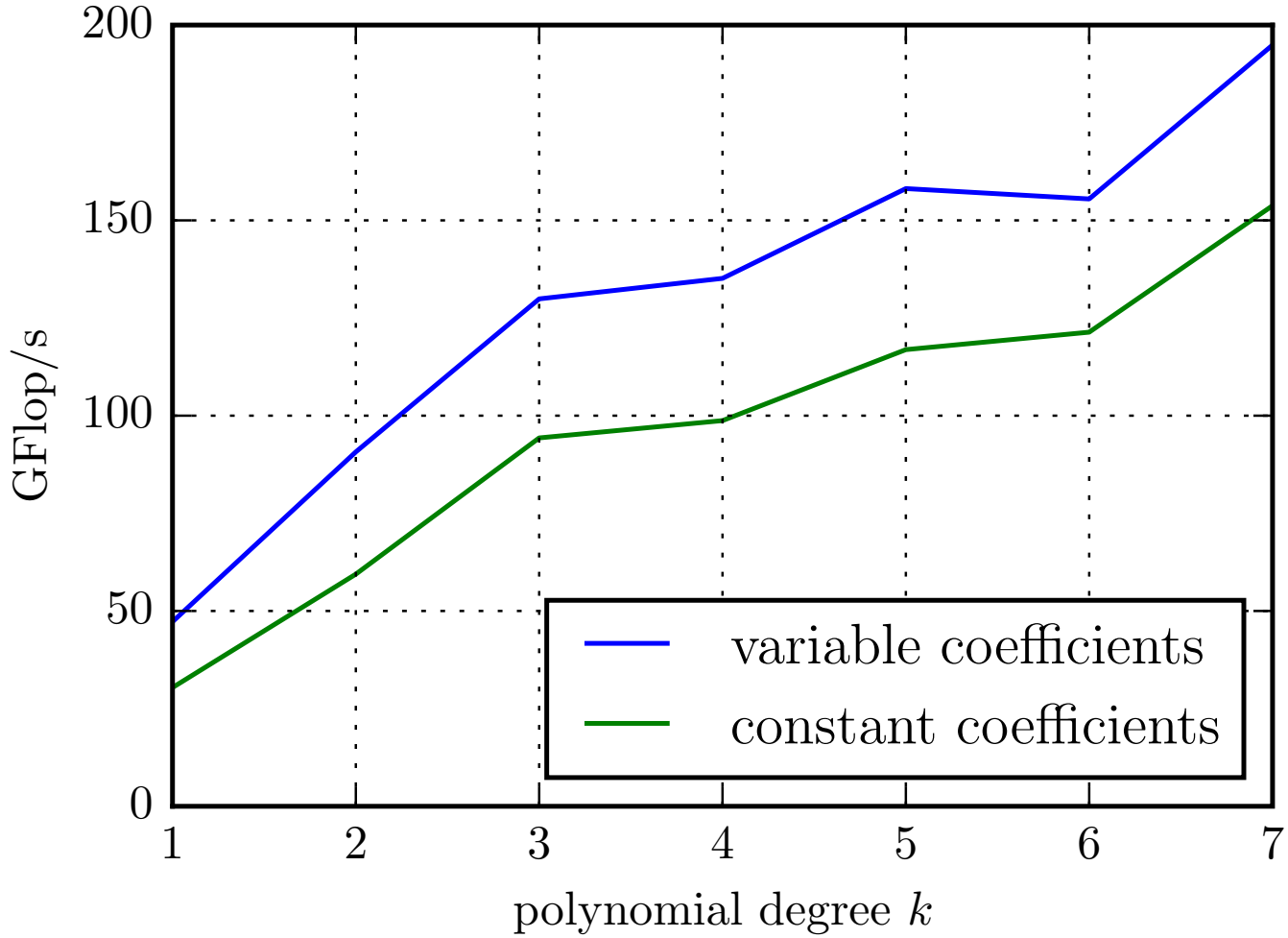
- **All results by Steffen Müthing**
- SIPG for 3d stationary convection-diffusion operator
- Matrix-based uses blocked SELL-C-σ matrix format
- Matrix-assembly is sum-factorized!

GFlop Rates for Volume Term



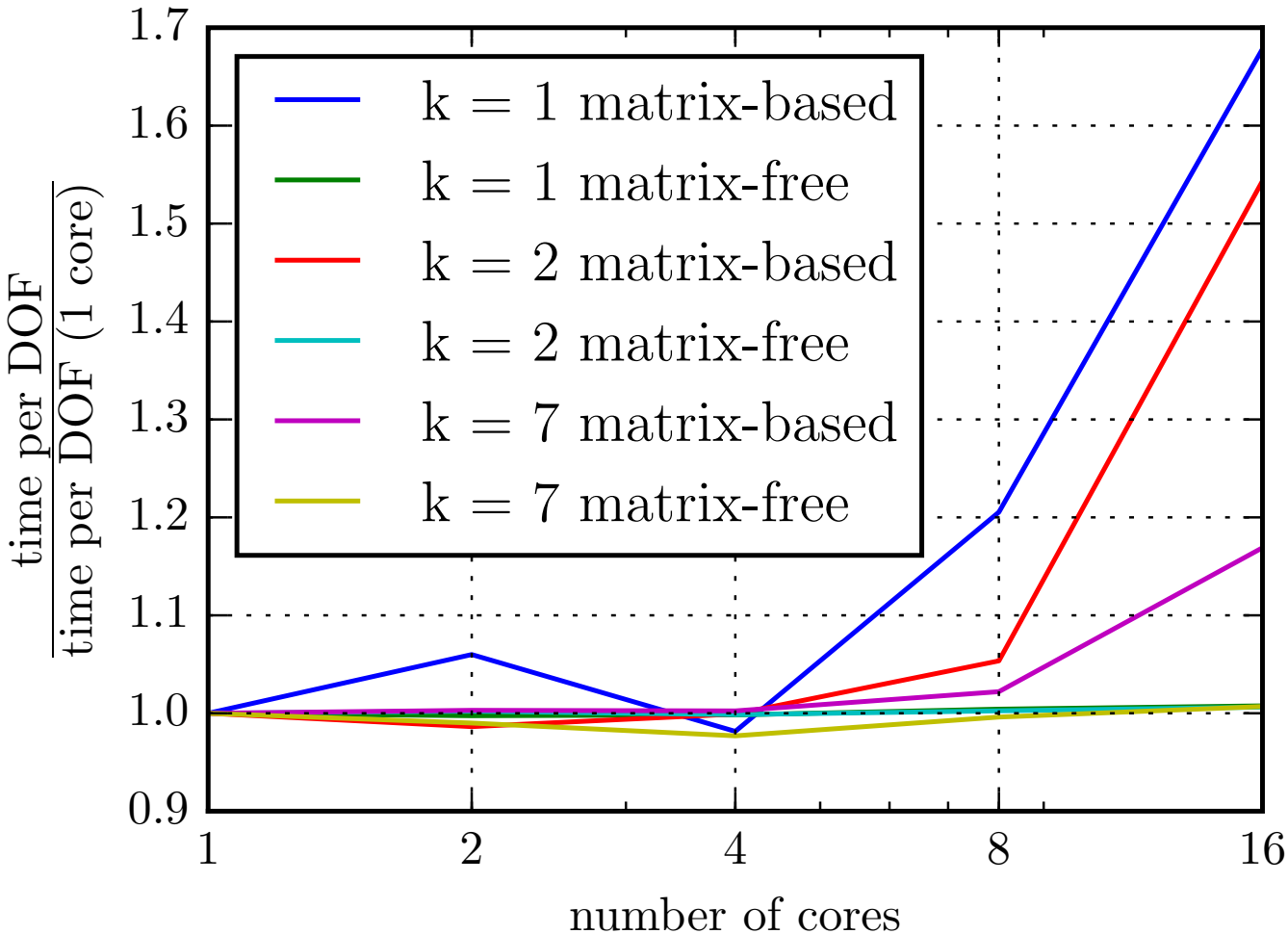
- SIPG for 3d stationary convection-diffusion operator
- Xeon E5-2698v3 (Haswell), Peak $\approx$ 40 GFlop/s per core
- Variable diffusion tensor and velocity field

GFlop Rates on One Socket



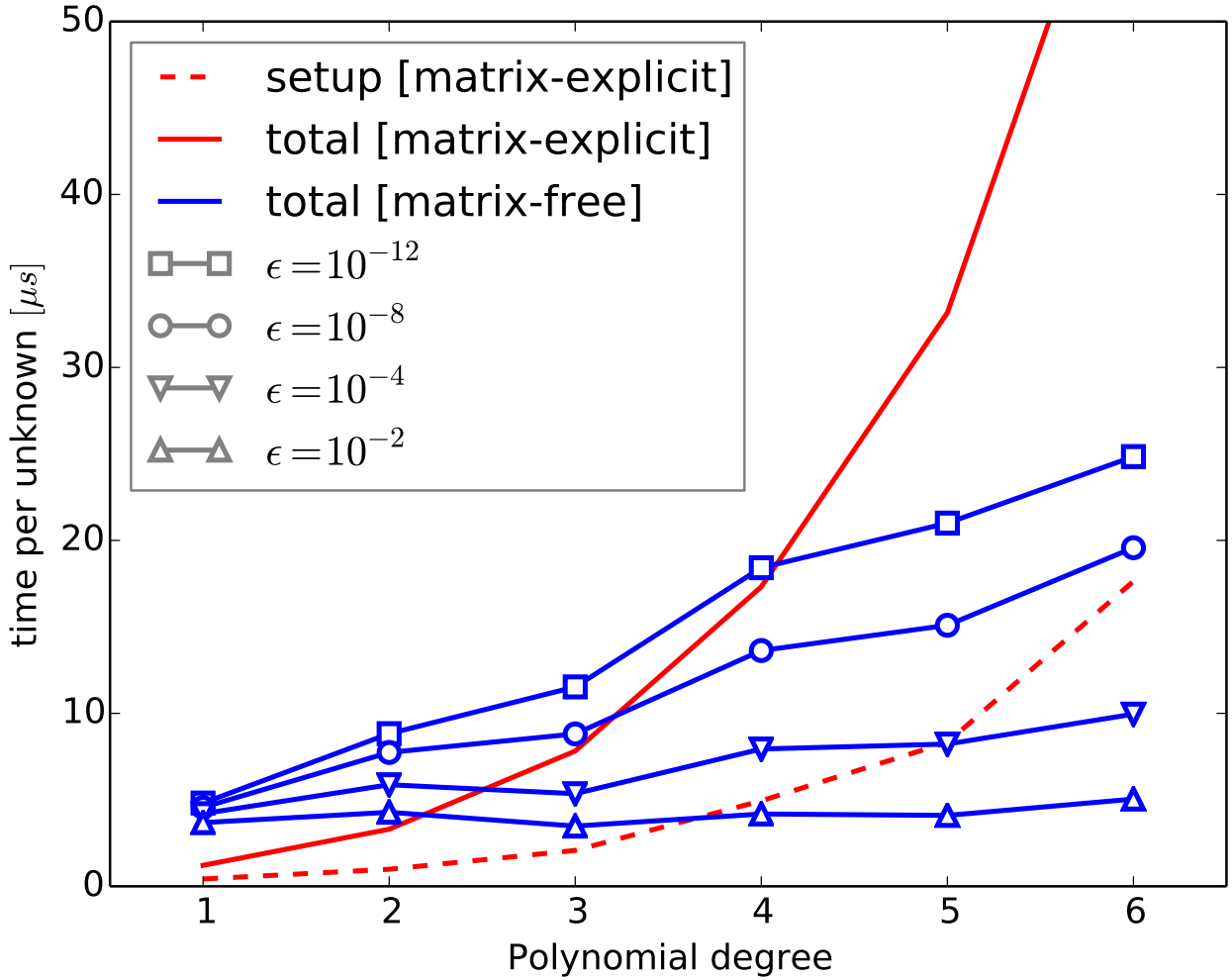
- SIPG for 3d stationary convection-diffusion operator
- Xeon E5-2698v3 16 cores (Haswell), Peak $\approx$ 650 GFlop/s
- Variable diffusion tensor and velocity field

Shared Memory Scalability



- Weak scaling: Constant #DOFs per core
- Xeon E5-2698v3 16 cores (Haswell)
- Matrix-based uses blocked SELL-C-σ matrix format

Matrix-free Block-Jacobi



- **All results by Eike Müller**
- SIPG for 3d stationary convection-diffusion equation

Contents

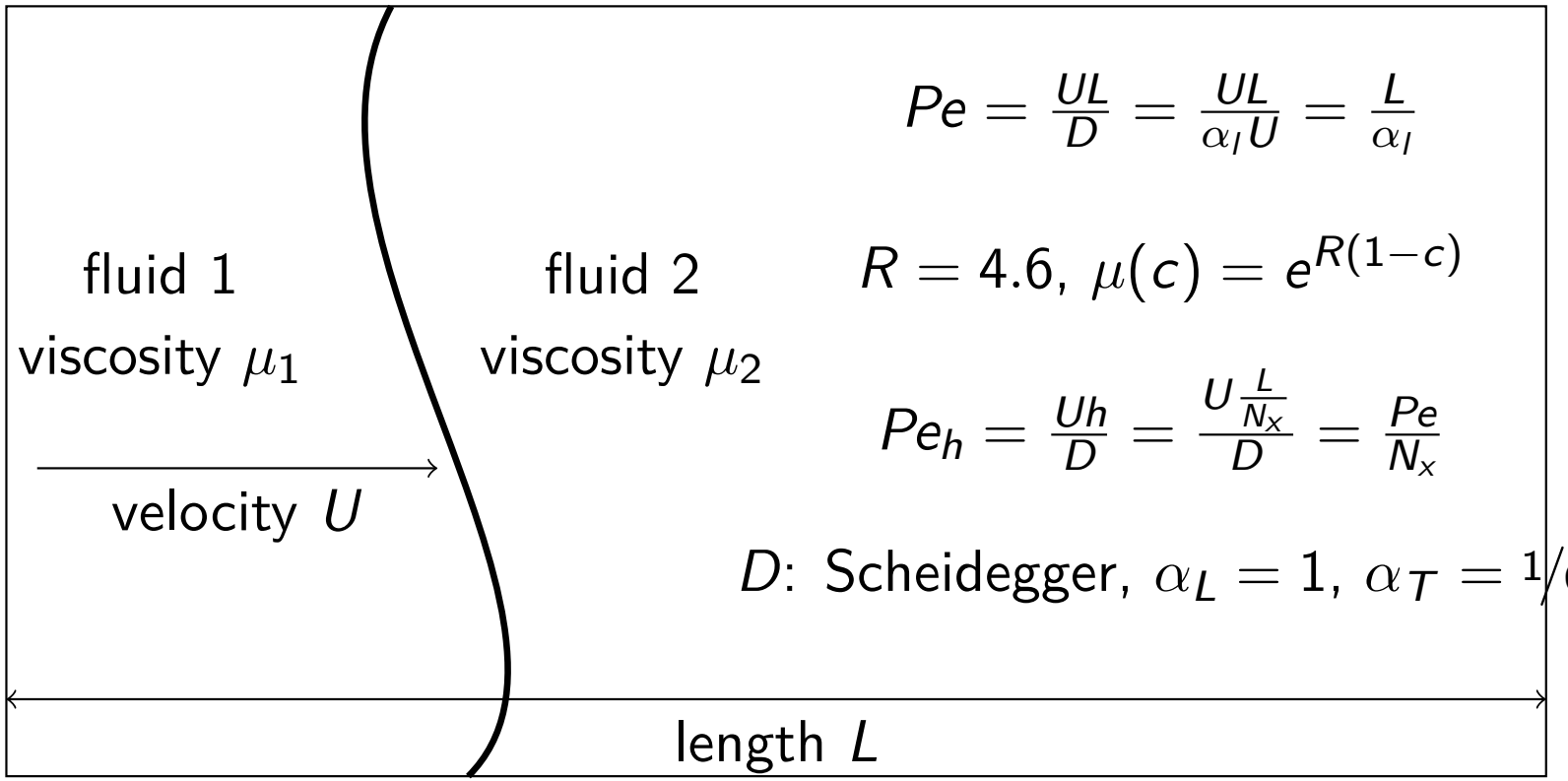
Miscible Displacement: Governing Equations

$$\nabla \cdot v = 0, \qquad v = -\frac{K}{\mu(c)}(\nabla p - \rho g)$$

$$\partial_t(\Phi c) + \nabla \cdot (cv - D(v)\nabla c) = 0$$

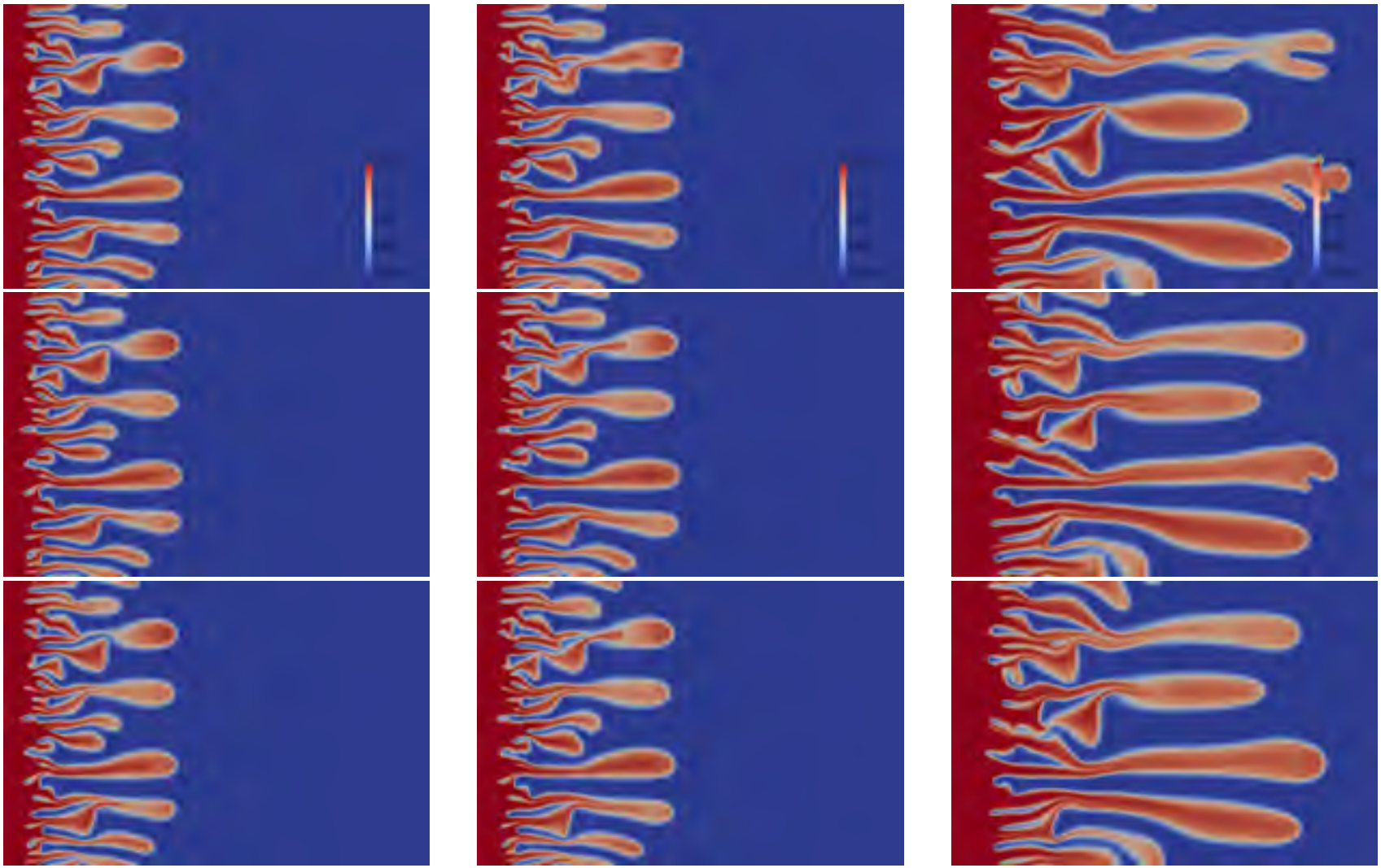
- Application e.g. in polymer flooding
- Unstable for less viscous fluid displacing more viscous fluid
- Decoupled implicit pressure / implicit concentration algorithm

Miscible Displacement: Governing Parameters



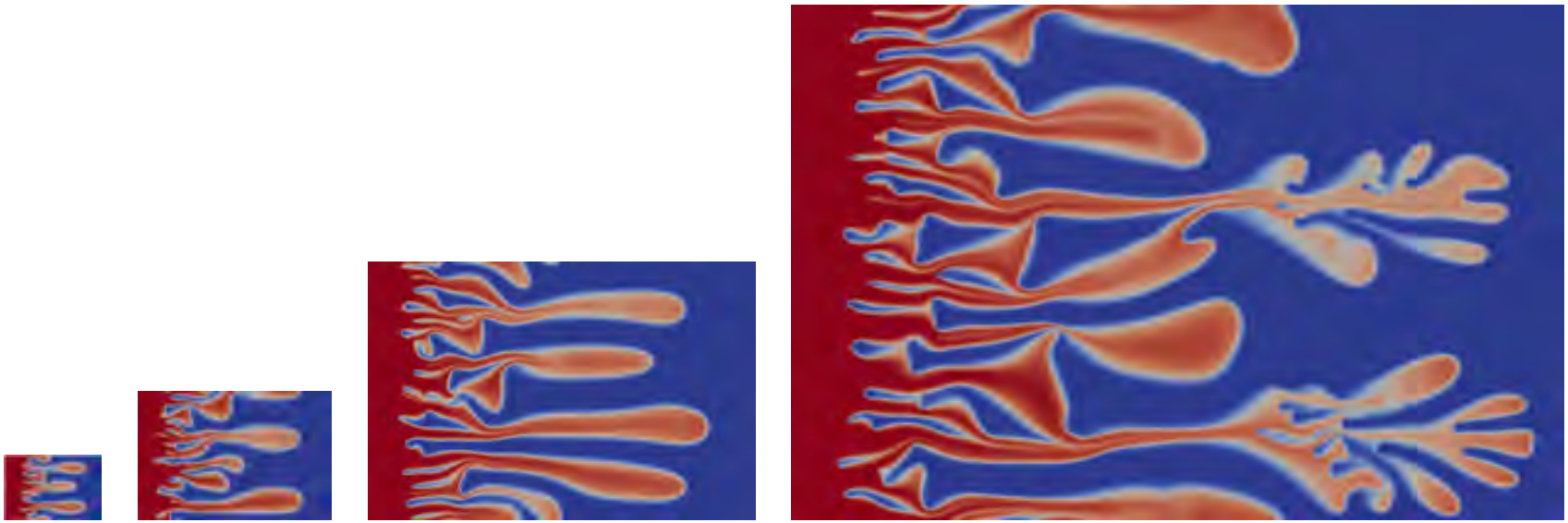
- Unstable flow for $R > 0$
- Need $Pe_h \approx 1$ to avoid unphysical oscillations
- Aspect ratio $A = 3 : 2$, periodic in y

Grid Convergence, $\mathbf{Pe} = \mathbf{1800}, \mathbf{R} = \mathbf{4.6}$



Top to bottom: $Q_1/Q_1, Pe_h = 2,\ Q_1/Q_1, Pe_h = 1,\ Q_2/Q_2, Pe_h = 2$
(All DG methods)

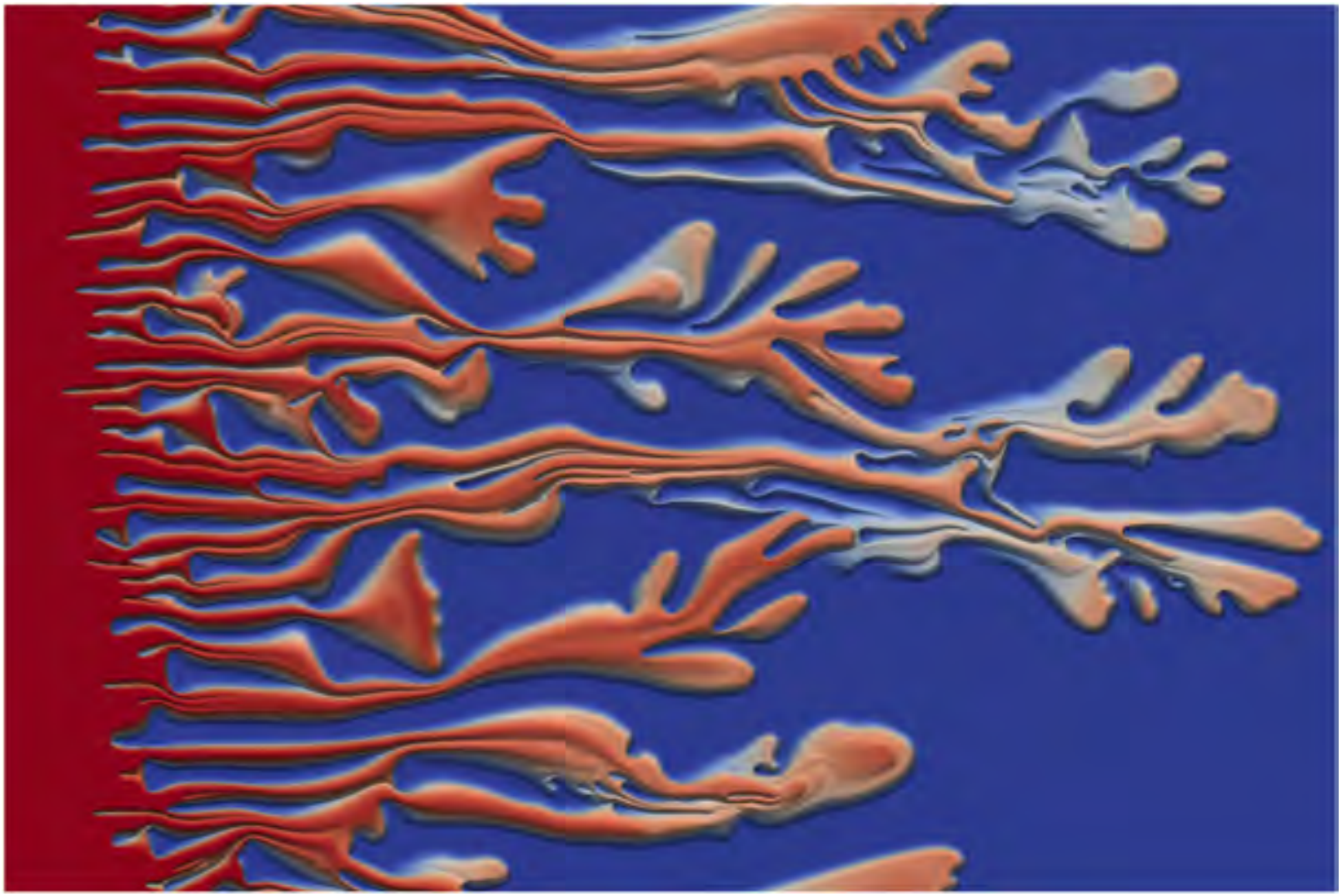
Increasing Peclet Number



$Pe = 450$ $Pe = 900$ $Pe = 1800$ $Pe = 3600$

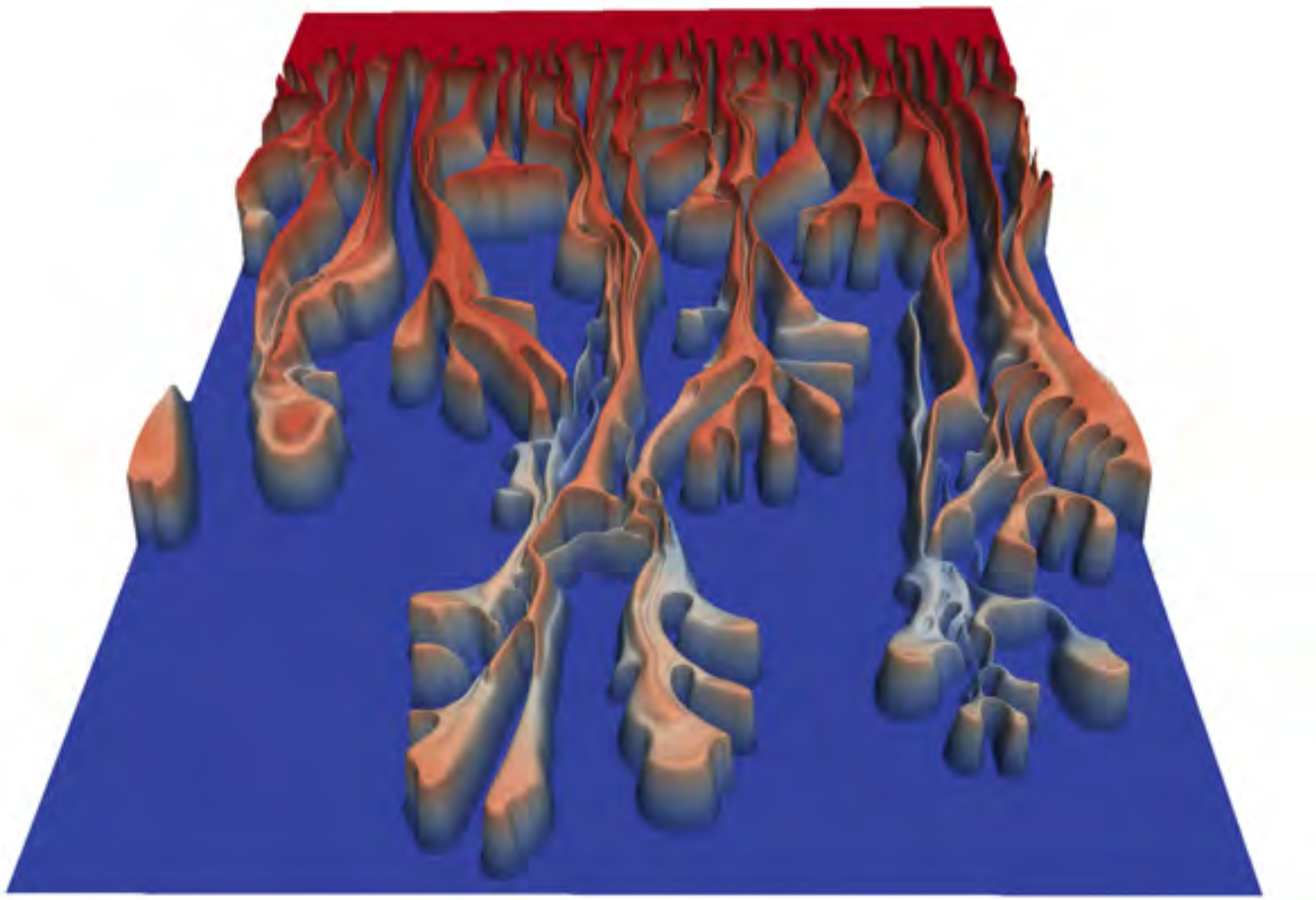
- Increasing Pe corresponds to increasing L and time
- Initially, fingers compete and merge, leading to spikes running on top of existing fingers
- Tip splitting starts at $\approx Pe = 2000$
- Interaction of splitted fingers becomes increasingly complex

$\mathbf{Pe} = \mathbf{7200}$ Top View



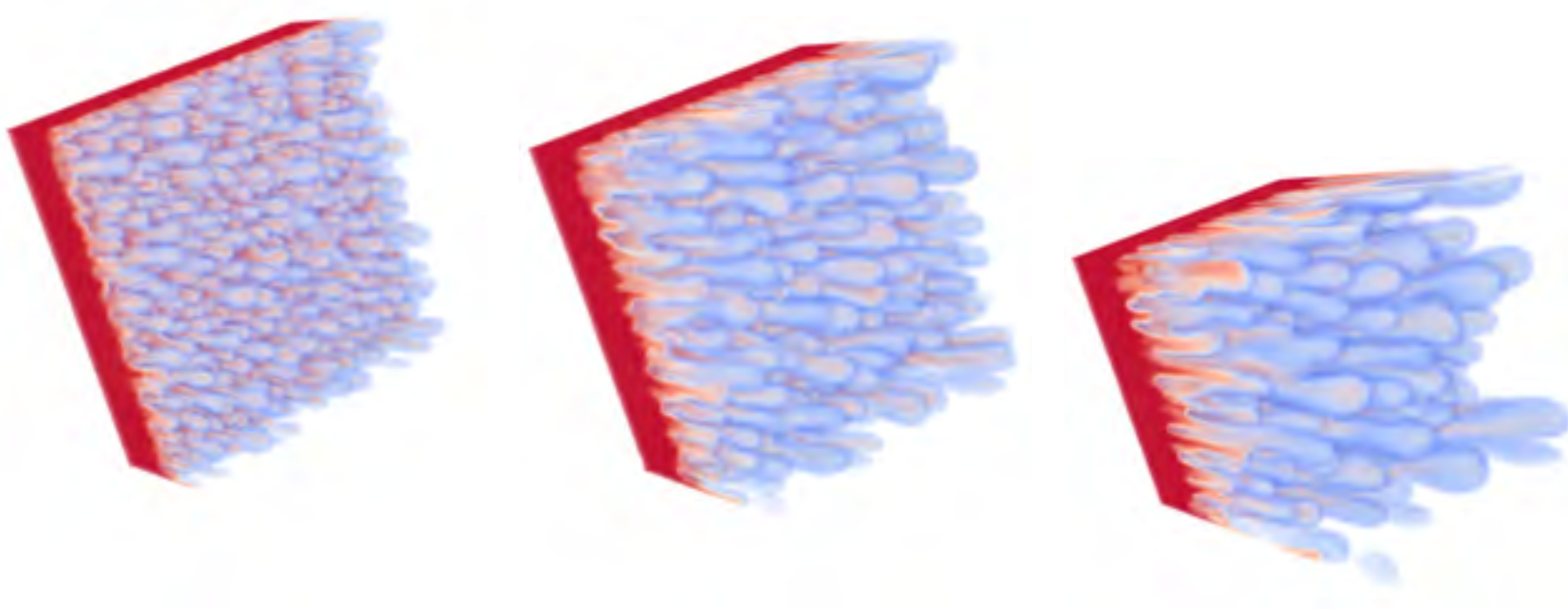
128 $\times$ Xeon E5-2630 v3 8 core, $155 \cdot 10^6$ DOF, Alexander-2, 17000 time steps, 32h

$\mathbf{Pe} = \mathbf{7200}$ Warped View



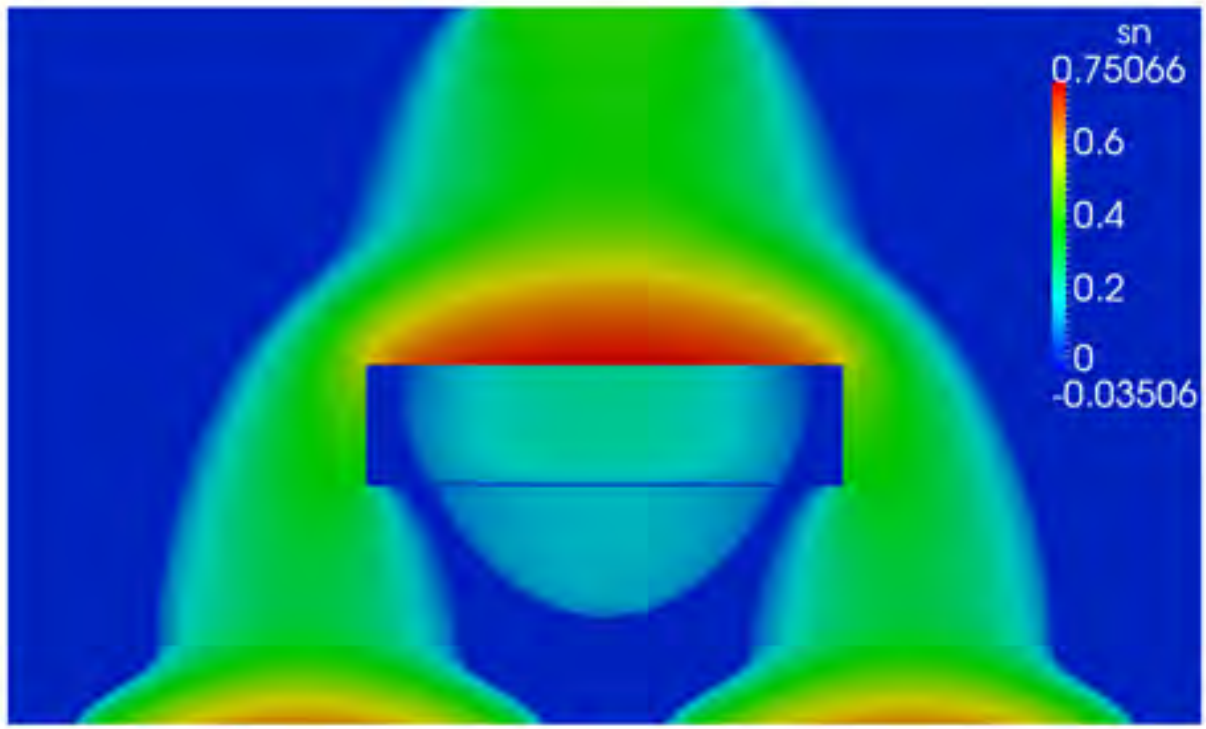
128 $\times$ Xeon E5-2630 v3 8 core, $155 \cdot 10^6$ DOF, Alexander-2, 17000 time steps, 32h

First 3d Results for **Pe = 300**



8 × Xeon E5-2680 v3 10 core, $243 \cdot 10^6$ DOF, Alexander-2

A DNAPL Infiltration Model Problem

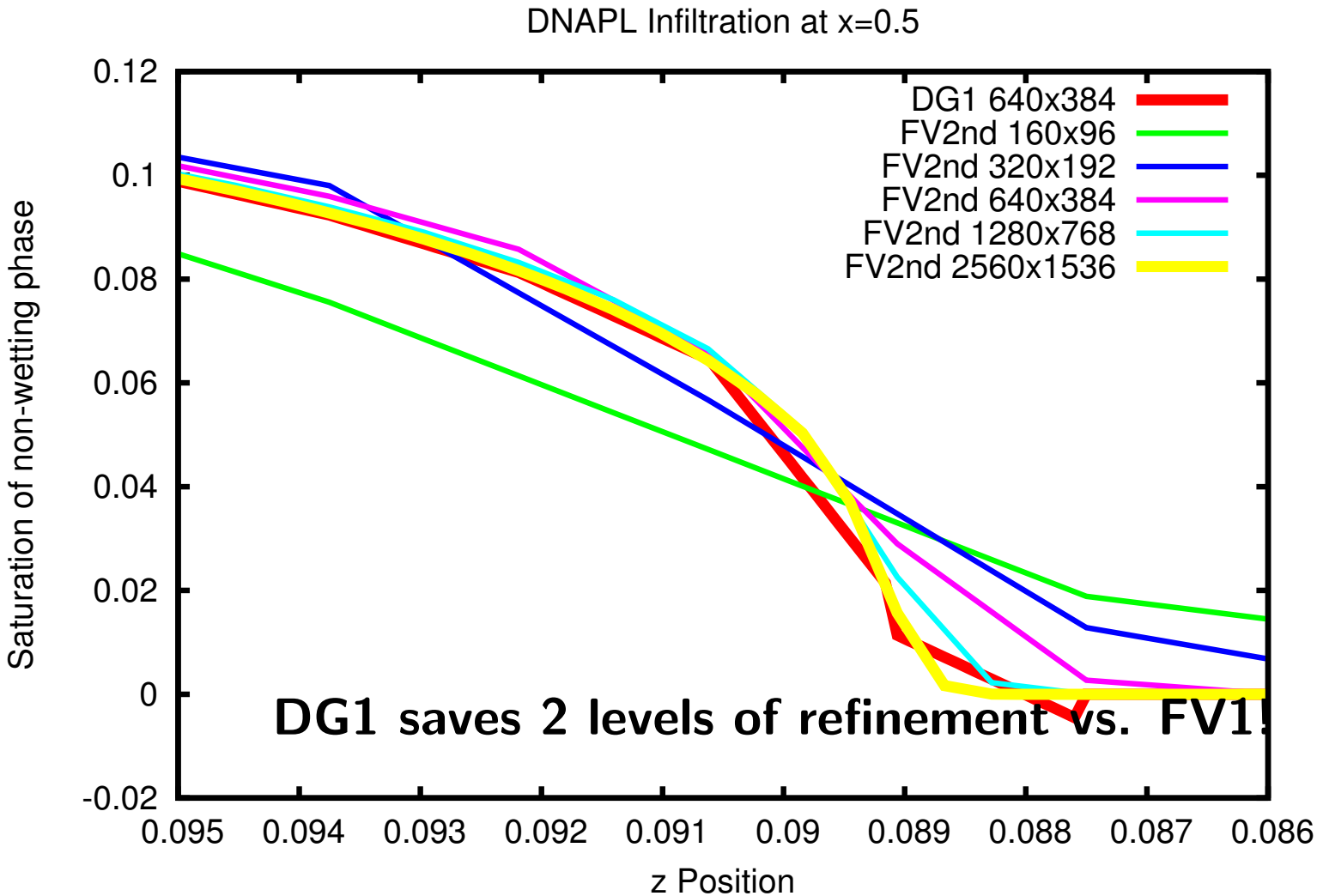


- $k_{r\alpha}$: quadratic
- p_c: Brooks-Corey, $\lambda = 2$
  $p_e^{bulk} = 755$,
  $p_e^{lense} = 1163$

- Fully implicit fully coupled
- DG1 / Alexander(2)
- CCFV0: full upw., impl. Euler
- CCFV1: central, Alex.(2)

At Free Boundary

DG1 640x384 vs. FV1



Weak Scalability for DNAPL Infiltration

$T = [0, 3600s]$, DG1/Alexander(2) vs. FV1/Alexander(2)

	P	h^{-1}	DOF	TS	ANL	ALIN	TT	LTIT
DG1	1	40	$7.7 \cdot 10^3$	30	11.4	1.8	156.8	0.144
	4	80	$3.1 \cdot 10^4$	71	11.0	3.4	466.1	0.052
	16	160	$1.2 \cdot 10^5$	125	12.0	5.7	1187.3	0.062
	64	**320**	$\mathbf{4.9 \cdot 10^5}$	**263**	**11.4**	**9.1**	**3250.5**	**0.067**
	256	640	$2.0 \cdot 10^6$	563	10.7	15.3	11233.0	0.082
	1024	1280	$7.9 \cdot 10^6$	1369	9.2	17.0	41917.4	0.112
FV1	1	160	$3.1 \cdot 10^4$	60	7.7	1.8	191.2	0.043
	4	320	$1.2 \cdot 10^5$	120	8.2	2.2	592.1	0.069
	16	640	$4.9 \cdot 10^5$	246	8.4	2.6	1453.7	0.078
	64	**1280**	$\mathbf{2.0 \cdot 10^6}$	**491**	**8.9**	**3.3**	**6496.1**	**0.151**
	256	2560	$7.9 \cdot 10^6$	1021	9.7	4.5	12774.9	0.156

Accuracy(DG1) = Accuracy(FV1) on 2 times refined mesh

Summary

- Agglomeration Algebraic Multigrid is reasonably robust and scalable
- Sum factorization has reduced complexity at increased performance
- Hybrid AMG-DG preconditioner can overcome memory wall and improve scalability
- All methods presented in this talk have been implemented in the DUNE software framework (`www.dune-project.org`) as part of the **EXA-DUNE** project
- DUNE is also the basis of the *Open Porous Media Initiative* driven by StatOil (`http://opm-project.org`)

Thank you for your attention!